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Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning

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ABSTRACT

The purpose of this paper is to further the notion of defining as a mathematical activity by elaborating a framework that structures the role of defining in student progress from informal to more formal ways of reasoning. The framework is the result of a retrospective account of a significant learning experience that occurred in an undergraduate geometry course. The framework integrates the instructional design theory of Realistic Mathematics Education (RME) and distinctions between concept image and concept definition and offers other researchers and instructional designers a structured way to analyze or plan for the role of defining in students' mathematical progress.

1. Introduction

Mathematical definitions should single out a concept with certainty, be minimal and elegant (Van Dormolen & Zaslavsky, 2003; Vinner, 1991), capture or synthesize the mathematical essence of a concept (Borasi, 1991), and serve as links in deductive chains of reasoning (Freudenthal, 1973). These statements point to preferred characteristics of mathematical definitions (e.g., minimal, elegant) and to the function they serve (e.g., single out the concept with certainty, serve as links in deductive reasoning). In order to emphasize the functional aspects of mathematical definitions, we use the term defining rather than definitions and cast defining as a mathematical activity (Freudenthal, 1973; Mariotti & Fischbein, 1997; Rasmussen, Zandieh, King, & Teppo, 2005). The purpose of this paper is to further the notion of defining as a mathematical activity by elaborating a framework that structures the role of defining in student progress from informal to more formal ways of reasoning. The framework we develop contributes to the field by offering other researchers and instructional designers a structured way to analyze and/or plan for the role of defining in students' mathematical progress.

We developed the defining as a mathematical activity (DMA) framework to account for a significant learning experience that involved creating and using definitions that occurred in a classroom teaching experiment. Our retrospective account that resulted in the DMA framework integrates two important lines of research: the instructional design theory of Realistic Mathematics Education (RME) and distinctions between concept image and concept definition. In the next section we delineate the theoretical background for our work by reviewing the literature on these two lines of research and relating this literature to our perspective on defining.

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2. Theoretical background

2.1. The instructional design theory of Realistic Mathematics Education

Freudenthal (1983) the founding father of RME, described mathematical concepts, structures, and ideas as inventions that humans create to organize the phenomena of the physical, social, and mental world. A central RME heuristic that captures this spirit is referred to as "emergent models." The intention of the emergent model heuristic is to create a sequence of tasks in which students first develop *models-of* their mathematical activity, which later become *models-for* more sophisticated mathematical reasoning (Gravemeijer, 1999). In a similar way, we demonstrate in this paper that definitions can first come to the fore as a *definition-of* students' previous activity and later these definitions serve as tools for (that is, *definitions-for*) further mathematical reasoning.

The shift from model-of to model-for (or definitions-of to definitions-for) is compatible with Sfard's (1991) process of reification. While the literature offers several examples of emergent models, the precise meaning of the term "model" is left implicit. To add clarity to the field, we define models as student-generated ways of organizing their activity with observable and mental tools. By observable tools we mean things in their environment, such as graphs, diagrams, explicitly stated definitions, physical objects, etc. By mental tools we mean the ways in which students think and reason as they solve problems—their mental organizing activity. We make no sharp distinction between the diversity of student reasoning and the things in their environment that afford and constrain their reasoning.

The global transition from model-of to model-for can be further refined in terms of four layers of activity referred to as Situational, Referential, General, and Formal (Gravemeijer, 1999). These four layers of activity will be central in the DMA framework. Here we operationally define these four layers of activity and provide an illustrative example of each of the four layers from a first grade class learning to add single digit numbers (Gravemeijer, Cobb, Bowers, & Whitenack, 2000). In Section 4 we describe the analogous forms of activity in the case of defining.

In the first grade example, the models are student-generated ways of organizing their activity with a double-decker bus, the arithmetic rack and conventional symbolism. Situational activity involves students working toward mathematical goals in an experientially real setting. Students act out the situation of loading and unloading passengers on a double-decker bus to determine the total number of passengers. Referential activity involves models-of that refer (implicitly or explicitly) to physical and mental activity in the original task setting. Students slide beads on an arithmetic rack to indicate persons getting on and off each of the two bus levels and to calculate how many people are on the bus. Students' organizing activity with the beads functions as a model-of the relevant physical and mental activity in the bus setting.

General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. Students' organizing activity with the arithmetic rack functions as a model-for solving single digit addition and subtraction problems in ways that do not refer to the double-decker bus setting. Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity. Students reason about numeric relations in ways that reflect new place value understandings and consequently use conventional notation without having to interpret or unpack the meaning of those symbols in terms of the arithmetic rack.

We concur with Gravemeijer, Bowers, and Stephan (2003), that the model-of/model-for transition should be reserved for a significant conceptual transition in student activity, where activity includes mental activity. Otherwise, each act of symbolizing would be labeled as a model-of/model-for transition and hence lose analytic power. The model-of/model-for transition is therefore concurrent with the creation of a new mathematical reality. In the first grade class the new mathematical reality that develops is the structured world of number relations. In our case of defining, the new mathematical reality is the world of triangles on the sphere, properties of these new objects, and relationships between these objects.

2.2. Concept image and concept definition

The term "concept definition" typically refers to the formal definition whereas "concept image" refers to the "set of all mental pictures associated in the students' mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356). One strand of studies that make use of this distinction details how students often reason from their concept image even though they "know" the definition (e.g., Edwards, 1997; Moore, 1994; Sierpinska, 2000; Vinner & Dreyfus, 1989). Other studies that make use of this distinction delineate students' concept images of particular mathematical ideas (e.g., Artigue, 1992; Rasmussen, 2001; Rasmussen & Ruan, 2008; Wilson, 1993; Zandieh, 2000; Zandieh & McDonald, 1999). Detailing concept images was also not limited to students. For example, Zandieh (2000) developed a framework describing the mathematical community's concept image of derivative and Cantoral (1989) characterized the historical evolution of the concept images of Taylor series.

One similarity between many of these studies is the juxtaposition and/or comparison (either implicitly or explicitly) of students' concept image and the intended mathematical meaning of a concept in terms of its definition. Such analyses typically focus on students' mathematical reasoning and whether students' concept images are compatible with the formal concept definition. A slightly different perspective on definitions involves examination of how students actually *use* and *create* concept images and concept definitions.

2.2.1. Using concept images and concept definitions

One strand of work that emphasizes student *use* of definitions can be characterized by what we call the "definition game." The definition game takes a variety of forms, including using an *unfamiliar* definition in a *familiar* mathematical setting and using a *familiar* definition in an *unfamiliar* mathematical setting. More generally, playing the definition game consists of using a ________ definition in a ________ mathematical setting, where either blank can be filled in with an adjective along the unfamiliar to familiar continuum. Variations of the definition game are present in our analysis and framework and hence we offer two illustrative examples. Our first example highlights the work of Dahlberg and Housman (1997), who investigated the strategies senior and junior undergraduate mathematics majors used when presented with a new definition in a relatively familiar setting. In particular, students were presented with the definition of a fine function, something that none of the students had previously encountered, and were asked to generate examples of fine functions, to decide whether given functions were fine, and to determine the validity of different statements about fine functions. We characterize this study as using an unfamiliar definition in a familiar mathematical setting because the setting for students' work with fine functions involved the relatively familiar notions of function and root (zero) in either the real or complex plane.

The second study we highlight is the work by Borasi (1991). In contrast to Dahlberg and Housman's study in which students used an unfamiliar definition in a relatively familiar mathematical setting, Borasi (1991) described a lesson with two secondary school students where they used a familiar definition of circle in an unfamiliar mathematical setting. The unfamiliar mathematical setting was that of taxicab geometry, which is an idealization of an urban area with a regular grid pattern of streets where distance is not the usual Euclidean distance but is measured only along the grid lines. Although street grids were familiar to these students, this environment was *mathematically* unfamiliar to students due to the way distances are measured.

2.2.2. Creating concept images and definitions

In comparison to research that focused on using definitions, other research focuses more directly on *creating* concept images and their definitions (e.g., Borasi, 1991; De Villiers, 1998; Furinghetti & Paola, 2000; Mariotti & Fischbein, 1997). De Villiers (1998), for example, reported on a teaching experiment in which secondary school students were provided opportunities to create definitions for geometric objects that were familiar yet no formal definition had been given in earlier classes. Mariotti and Fischbein (1997) highlighted the complexity of a defining process that involved creating both concept images and concept definitions for sixth graders who, after having some experience "unfolding" various solids to form a 2D representation (referred to as a "net"), made inroads to creating and comparing different definitions for a net.

As evidenced in many of the studies that address either *using* or *creating definitions*, engaging students in the defining process can provide opportunities for students to appreciate the arbitrariness of definitions and to make explicit characteristics of "good" and/or useful definitions (cf. Zaslavsky & Shir, 2005). In some instances, engaging students in creating definitions opens a need to define additional geometrical concepts, develop theorem-like conjectures about concepts, compare alternative definitions and revise definitions. As Lakatos's (1976) seminal historical analysis of polyhedron points out, the formulation of a definition and of what the community wants the concept of polyhedron to be, can evolve concomitantly in a process of "proofs and refutations." Larsen and Zandieh (2008) illustrate this at the undergraduate level with an example from abstract algebra.

2.3. What we mean by defining

At first glance it might seem relatively simple to indicate what is and is not a defining activity. For example, the most straightforward characterization of a defining activity would be "creating a definition." However, our review of the literature and analysis of our case study data suggests that such a straightforward characterization misses the fact that formulating a definition, negotiating what one wants a definition to be (and why), and refining or revising a definition can occur as students are proving a statement, generating conjectures, creating examples, and trying out or "proving" a definition. We therefore find it necessary to include these other types of activity as part of defining when they involve formulating, negotiating, and revising a definition. The DMA framework reflects this more nuanced view of what constitutes defining.

3. The classroom teaching experiment

The DMA framework is the result of a retrospective account of a classroom teaching experiment (Cobb, 2000) that we[1] conducted during a 5-week summer session with 25 students at a large Southwestern university in the United States. The student composition was quite varied, and consisted of nine computer science majors, three mathematics majors, one mathematics minor, five preservice mathematics teachers, and seven high school mathematics teachers who completed extra assignments unrelated to the sphere to receive masters level credit.

[1] The research team consisted of Michelle Zandieh, Chris Rasmussen, Barbara Edwards, and Libby Krussel. Zandieh's role was that of teacher and co-researcher, Rasmussen served as a "witness"(Steffe & Thompson, 2000) for every class session, and Edwards and Krussel made periodic visits throughout the teaching experiment.

The data collected included videorecordings of all class sessions using two video-cameras, individual interviews with 22 of the 25 students at the beginning and end of the term, copies of students' homework and exams, students' journal entries submitted twice a week, and students' portfolios consisting of mathematical reflections on their learning for self-selected homework problems.

The data analysis focused initially on Day 6 of class because we recognized something powerful was happening in terms of student learning on that day and we wanted to characterize it. Maher and Martino (1996a,b) refer to such occasions as "critical events." Such events involve conceptual leaps and hence demand attention and explanation. As we began to study this data we realized that the powerful event might be what Gravemeijer (1999) calls the creation of a new mathematical reality and that the emergent models heuristic with its four layers of activity might therefore be a way to describe how student reasoning was evolving. We also knew that this event was focused around defining and we conjectured that notions of concept image and concept definition were intimately involved.

As we examined the data for the four levels of activity, we began an iterative cycle of refining our understanding of the data commensurate with developing a way to interpret the emergent models heuristic for the case of defining, reviewing the literature, and refining our conjecture on the role of concept images and concept definitions. During this process we realized that the relevant student activity extended to Days 7 and 8 and even to one day much later in the class (Day 22) to capture Formal activity. This analysis cumulated in the development of the DMA framework.

The course used the text *Experiencing Geometry on Plane and Sphere* (Henderson, 1996). We found the sequence of problems in this text to be compatible with the RME perspective that mathematics is a human experience of organizing subject matter at one level and then progressing to more conventional or formal levels. At the heart of this perspective is that students create meaningful mathematical ideas as they engage in challenging tasks.

Instruction generally followed an inquiry-oriented approach and classroom interactions fell into three main categories: whole-class discussion with the teacher in front of the class, whole-class discussion with a student in front of the class, and students working in small groups. The whole-class discussion categories functioned in a variety of modes: a mini-lecture, with one person presenting material and perhaps taking brief questions or eliciting brief comments from the class; a true classroom discussion moderated by the teacher but consisting mostly of student comments, often from many different individuals; question and answer period—teacher responds with short answers to student questions, usually without discussion, especially regarding course logistics such as dates and expectations for tests or homework assignments.

4. The defining as a mathematical activity framework

The DMA framework is a way of highlighting the role that defining can play in students' transition from less formal to more formal ways of reasoning, in particular in ways that create a new mathematical reality for those students. The framework brings together the four levels of activity from RME with the notions of creating and using concept images and concept definitions. The particulars of the framework describe the relationships we found as we analyzed the data from our students' creation of the new mathematical reality of triangles on the sphere. They also create a starting point for other researchers analyzing the role of defining in student learning. In addition, they illustrate the role that defining may play for curriculum designers interested in having students develop formal mathematics from informal starting points.

Table 1 highlights the relationships inherent in the DMA framework. In particular we note that Situational activity involves using a concept image to create a concept definition, Referential activity focuses on using a concept definition to create a concept image, General activity involves creating new concept images and concept definitions, and Formal activity focuses on using established concept images and concept definitions to serve other mathematical goals.

In this section of the paper we lay out in detail a paradigmatic example of how students experience the creation of a new mathematical reality with an emphasis on the role of defining in this process. Following the notion of emergent models from RME, we see definitions play the role of "model" and so there is a transition from students creating a definition-of that describes their reasoning to them using a definition-for more formal reasoning.

The section is organized into subsections on Situational activity, Referential activity, General activity and Formal activity. These form a chronological narrative of student activity in the geometry course. For each level of activity, we analyze classroom transcripts to highlight the role that creating and using concept images and concept definitions play in each of the four levels of activity. The most important aspects of this transition from less formal to more formal activity occur in

Table 1
The defining as a mathematical activity framework.

Theoretical construct	Four levels of activity			
	Situational	Referential	General	Formal
Creating a concept definition	x	–	x	–
Using a concept definition	–	x	–	x
Creating a concept image	–	x	x	–
Using a concept image	x	–	–	x
Creating a new mathematical reality	–	x	x	–

Note: lowercase x indicates a high incidence of a construct in a given level of activity, whereas – indicates a relatively lower incidence.

the subsections on Referential and General activity. Within these subsections we will also bring to the fore aspects of the definition-of/definition-for transition related to the creation of a new mathematical reality.

4.1. Situational activity: creating a concept definition using a concept image

In this section we trace students' mathematical work as they engaged in the task of creating a definition for a triangle on the plane using their rich concept images of triangle. In previous tasks students had discussed the meaning of straight line and created definitions for angle, and so creating a definition for triangle was not an unusual or altogether unexpected task. Students first discussed their ideas in small groups for approximately 10 min, wrote their definitions on the board, and then participated in a whole-class discussion that lasted approximately 30 min.

Although planar triangles were not previously discussed in this class, students' past experiences with triangles were sufficiently rich that they were able to immediately begin to create a definition. In terms of RME levels of activity, we characterize students' reasoning surrounding this task as Situational because students' rich concept images of planar triangle afforded them progress on their own accord. In other words, planar triangles were experientially real mathematical objects for these students. This does not mean that the definitions they created were issue-free. Indeed, as we subsequently illustrate, defining planar triangle involves much more than simply stating or writing down a definition. It also involves revising or refining a definition mediated by negotiations about what wording should be used, negotiations of what the criteria for judging the adequacy or acceptability of a definition should be, and even negotiations for what the concept itself should be.

4.1.1. Small group discussion

In the two small group discussions for which we have video data (subsequently referred to as the front group and back group), students acted as if their concept image of planar triangle was consistent with that of the others in their group. For example, after they had formulated a definition they were comfortable with (which took a few iterations of committing words to paper, discussing, and revising), Jay and Amy in the back group commented:

Jay: I think we already had in our mind a fixed idea of what a triangle is. We were just trying to [*Amy*: Trying to recreate the picture.] Yeah, with a clearer, simple definition.

Jay's comment, "yeah, with a clearer, simple definition" was indicative of both the front and the back group's concern that the definition they formulated be simple and understandable. Their concerns for simplicity and understandability included the need to avoid using previously undefined terms and decisions as to whether to include statements like "the sum of the interior angles equals 180 degrees" in their definition.

For example, in the front group after several iterations of formulating their definition, which at this point was stated as "an enclosed figure consisting of exactly three straight sides," Roxanne asked her group mates if it was necessary to say that a triangle has three angles. As illustrated in the following excerpt, she followed up her question with an addition to their definition, which then got revised due to Pete's suggestion that the angle measure sum is "something that automatically happens."

Roxanne: I added to mine. I said an enclosed figure with three straight lines that contains three angles whose angle measures add up to 180.
Russ: Say that again?
Chip: Interior angle measures.
Pete: Yeah. Do you have to make that a condition when it's something that automatically occurs? And then how do you prove that they add up to 180 degrees?
Roxanne: Well, do we have to say that the interior angles add up to 180 because that's a property of a triangle? [*Pete*: Exactly.] So maybe we can stop with a closed figure made with three straight lines containing three angles.

Apparently, Pete's suggestion dissuaded Roxanne from including the angle sum in the definition because "that's a property of a triangle." In the back group, Tom explained why he did not want to include in their definition the angle sum statement in the following way:

Tom: Well, you know, you definitely have [in your definition] the properties, not things that come from those properties. I think the properties are like the 3 sides and the 3 angles. [*Cindy*: The characteristics of a triangle.] Right, the angles equaling 180 seem to be derived from those properties. I don't know. That's why I won't include the 180 degree one. That's kind of the way I think of it.

The point we draw from these excerpts is that even when these students used a well-developed concept image to create a definition, their activity involved the mathematically deep issue of negotiating criteria for why certain elements (e.g., sum of the angle measures) should or should not be part of the definition. The significance of this type of activity for students is that it can contribute to developing a meta-understanding of mathematical definitions, what Edwards (1999) refers to as a concept image of definition.

While students sought simplicity in creating their definition, they were also aware of possible drawbacks of a definition that was too simple. For example, in the front group there was tension between wanting their formulation to be simple and a concern that it not be so simple that it excluded geometric objects that they view or consider to be triangles.

Pete: [in response to a proposed formulation by Roxanne] Okay, but, do you have to have endpoints? [sketches three rays that intersect to form a triangle] Is that not a triangle? Can you form a triangle with rays?
Georgia: But that's not the most simple, is it?
Pete: No, but you don't want to limit your definition.

In addition to wanting their definition to include all instances of triangles, students also wanted to formulate their definition so that it did not include geometric objects that were not, in their opinion, examples of triangles. For example, in the back group Amy suggested that they could define a triangle as the area contained in the intersection of 3 lines. Shortly thereafter Cindy pointed out that two parallel lines with a transversal fit Amy's description but was not a figure they wanted to call a triangle.

The sentiment that whatever formulation for triangle was finally agreed upon must separate instances of triangles from non-instances was expressed well by Jay.

Jay: You have to be careful that when you put it up there, somebody can't come up with a figure applying to your definition, that your definition applies to the figure that isn't a triangle. I mean you have to make sure there's no weird cases out there where it's not a triangle. You cover everything, but not more than you need.

Although we do not elaborate on the specifics, negotiating the wording of a definition so that it separates instances of the concept from non-instances was also a topic of conversation in whole-class discussion. Next, we describe another significant topic of conversation in whole-class discussion—whether or not (and why) one should consider extreme cases as triangles (e.g., figures with collinear vertices).

4.1.2. Whole-class discussion

In contrast to the two videorecorded small group discussions in which students acted as if they had compatible ideas for what is and what is not a triangle, the whole-class discussion brought out the fact that another group of students disagreed on whether or not a figure with all three vertices collinear (or coincidental) should be considered a triangle. Greg and Adam summarized for the class the disagreement in their group as follows:

Greg: I'm saying that it doesn't matter if the points are different, or if they happen to lie on the same spot or if they happen to lie on the same line. It doesn't make any difference, it's still a triangle, as long as you have 3 points and they're connected.
Adam: So he's saying that if we have three points on a line, it is still a triangle. Because you can connect one, you connect the other and then you connect the end two, but it's still the same line. It doesn't make sense.

We interpret Adam's statement, "It doesn't make sense" as an indication that what Greg is proposing conflicts with what he thinks should be a triangle. That is, it challenges his concept image of a triangle. As the discussion continued several other students expressed similar unease and/or uncertainty with Greg's extreme cases as examples of triangles.

One way in which students' uncertainty with the collinear vertices case expressed itself was in terms of whether or not such a figure would have interior angle sum of 180 degrees. Greg argued that this is in fact the case because "the two endpoints have 0 degrees, the center point has 180 degrees. It still equals 180." A figure similar to that shown in Fig. 1 was then sketched on the board by the teacher to illustrate Greg's argument.

Another way in which a student argued against the collinear vertices figure centered on the fact that the main engineering principle behind a triangle is that it is a rigid figure, and therefore the object in Fig. 1 should not be considered a triangle. The point of the previous discussion about extreme cases is that even for Situational activity in which triangles are experientially real for students, their defining activities can give rise to a whole host of concerns and issues requiring negotiation, including potentially re-thinking the very concept itself.

4.1.3. Summary

In Situational activity our analysis highlighted important mathematical issues and concerns that arose for students as they formulated, negotiated and revised definitions for planar triangle. In terms of RME layers of activity, we described students' activity as Situational because planar triangle is a geometric object for which students have rich concept images and therefore were able to immediately engage in creating a definition. Creating this definition, however, involved much more than simply expressing in words their concept image. It involved iterative refinements mediated by emerging sensitivities to criteria for what makes a "good" mathematical definition, including a need for a definition to be simple, to separate agreed upon instances from non-instances of triangle, and to exclude ambiguous or undefined terms. Students' Situational activity also included negotiations about the concept itself, as exemplified by the debate focusing on extreme cases. As indicated in Table 1, the focus of Situational activity is on using a concept image to create a concept definition. However, that does not

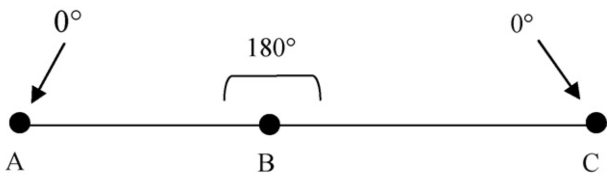

Fig. 1. A triangle with collinear vertices.

mean that there will not also be some student activity related to creating a concept image as with student decision-making about extreme cases. The next section on Referential activity focuses much more on the creation of a new concept image.

4.2. Referential activity: using a concept definition to create a concept image

In Referential activity students use the planar triangle definition (slightly modified for the sphere), along with the properties that they associate with this definition, to create examples of spherical triangles and to notice some of their properties, i.e., to begin to create a concept image of spherical triangles. Students' organizing activity with the definition and associated concept images of planar triangle applied to the sphere functions as a model-of (or definition-of) the relevant physical and mental activity in the plane. Using this definition to create a concept image of spherical triangles is the beginning of the creation of the new mathematical reality of triangles on the surface of the sphere. This process of exploring the properties of triangles on the sphere will continue and be expanded in the creating of concept images and concept definitions that occurs in the next section in General activity. In Referential activity the focus is on using the definition-of the Situational activity to create the new concept image. In other words, Referential activity is activity that continues to look back at the Situational activity involving triangles on the plane for similarities and differences between the two.

In Situational activity, students were creating the statement of a definition and exploring examples in the service of negotiating and refining their statement. Here we see students focused on interpreting a definition in a new setting. This is an example of the definition game that we discussed in the literature review. Students were interpreting a *familiar* definition in an *unfamiliar* setting. The familiar definition is the one for planar triangle they had just created, "A triangle is a closed figure formed by three straight line segments. Each segment intersects each of the other two segments only once at an endpoint." They were encouraged to modify this definition only by interpreting the phrase "line segments" as segments of great circles. The setting in which they were interpreting the definition is relatively unfamiliar. They had explored straight lines (i.e., great circles) and angles on the surface of the sphere, but had not previously considered triangles.

In reference to Table 1, the primary activities of the students at this level consisted of (1) creating a concept image by constructing potential examples of triangles (from the definition and their concept images of triangles on the plane), (2) using a concept definition (and sometimes concept images from the plane) to check whether or not the triangles were triangles and (3) creating a concept image by noting new properties that specific spherical triangles have (in comparison with planar triangles), and conjecturing properties that might be true for all triangles on the surface of the sphere and which are different from corresponding properties for planar triangles.

Sample transcripts will illustrate these activities from the second half of Day 6, the class period reported in the section on Situational activity. The data we are drawing from for Day 6 is a 20-min small group discussion followed by a 20-min whole-class discussion. The small group discussion began with the teacher's request for the students to find examples of triangles on the sphere that fit the "same" planar definition.

4.2.1. Small group work

In the back group, Cindy's initial reaction was to draw a triangle on the surface of the sphere that looks very much like a planar triangle by drawing three short line segments appropriately connected. Amy reacted, "Now draw a weird triangle."

Tom: These have to be straight lines right? They have to be part of a great circle.
Amy: Right, part of a great circle. [She extended Cindy's line segments to complete great circles.]

Note that Cindy seemed to use a prototypical image of triangle on the plane for her first drawing. Tom and Amy clarified by referring to a property of triangles that is also part of the reinterpreted definition of triangle for spherical triangles.

Sam also began the problem by drawing three complete great circles and then caught the attention of others in his group by creating an "ugly" triangle (see Fig. 2).

Sam: This is an ugly triangle. It starts here and it bows way out and then it terminates down there.
Cindy: Why isn't that the triangle [pointing to the smaller triangle that shares the lune/biangle[2] with Sam's triangle]?
Sam: They both are.
Amy: But that's not a triangle.
Tom: But it's only got two angles.
Sam: No, it's got three. One, two, three [pointing to each angle as he counts].
Tom: Oh, we weren't looking at the same thing. [Tom probably saw the biangle.]
Amy: Wait, I'm missing it.
Cindy: Oh, I get it, I get it, I get it.
Amy: Can I see the other side of that [sphere]?
Sam: It's not a traditional triangle, but it's correct by the definition.

We characterize Sam's contributions as Referential activity because he describes the spherical triangle both by referencing the definition of triangle developed on the plane (in its slightly modified form for the sphere) and by his reference to a difference ("ugly," "not a traditional triangle") with a concept image or prototypical image of (planar) triangle that he suspects that they all share. In addition to the definition, which is in terms of line segments, Tom and Sam used a property of

[2] A lune or biangle is a figure on the sphere with two vertices, two sides and two interior angles.

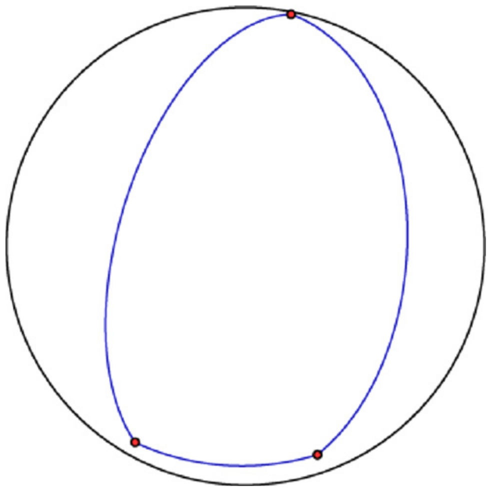

Fig. 2. Sam's triangle.

triangles that they were familiar with from the plane, the number of angles, as a way to negotiate whether or not something is a triangle. The confusion and excitement exhibited by the students is indicative of the beginning of creating a new mathematical reality. The conversation continued with a further discussion of Sam's triangle.

Amy: They're both obtuse angles. Two obtuse angles and an acute angle.
Jay: The sum of the degrees is greater than 180 then.
Cindy: But is that all right for a sphere?
Chris: That's real weird, huh?
Sam: Yeah.
Jay: But it works. [pause] Can we do another weird one? That was neat.

Jay's statement that "it works" is a further indication that the group was using the definition as the arbitrator of what would be considered a triangle, even though the triangle was "weird" (i.e., it did not fit their concept image of triangle from the plane). Amy and Jay also noticed properties that this triangle has that are different from triangles on the plane. Cindy wondered if the fact that the figure does not satisfy the angle sum property of triangles on the plane means that the figure is not a triangle on the sphere, while Jay seemed to indicate that it is a triangle by the definition. The activity in this discussion is Referential because it involves students organizing their activity on the sphere by referencing similarities and differences between triangles on the surface of the sphere and triangles in the plane. The activity also provides examples of students using a concept definition as part of negotiating of their incipient creation of spherical triangles.

4.2.2. Whole-class discussion

The whole-class discussion started off with examples from Betty (from another group) and Pete (from the front group) showing that one can construct eight triangles with three complete great circles, in particular, three great circles at 90 degrees to each other. In the next few minutes of whole-class discussion students generated a large number of conjectures and questions about this sum, some of which the teacher wrote on the board.

Amy: Does that mean that there are 270 degrees in every triangle?
Students: No, no.
Adam: I just think that 180 degrees only applies to the plane.
Teacher: [showing the whole class something a student has handed her on a triangle with three great circles enclosing eight numbered areas] She's telling me that this number 7 triangle here probably doesn't have 270 degrees for the sum of the angles.
Betty: But some of them have even more.
Jay: I'm wondering if you can get more than 270.
Pete: The sum of the angles is between 360 and 180 but not equal to either.
Betty: How do you know that?
Adam: Although Greg wants to dispute that it is equal to 360.
Wade: No, I would say between 180 and 270.
Adam: Can you get smaller that 180?
Jay: I think it's 270 also.

This conjecturing is Referential activity in that there are several ways in which students were creating their guesses and predictions based on their concept images of planar triangles. Amy and Adam at first discussed the sum of the interior angles as a constant, similar to that on the plane. Later student comments took a broader view but were still facilitated by a comparison to properties of triangles on the plane. In particular, students were focused on triangles that fit within a hemisphere and were created by using straight lines (great circles) to enclose an area as per their concept definition and concept image from the plane.

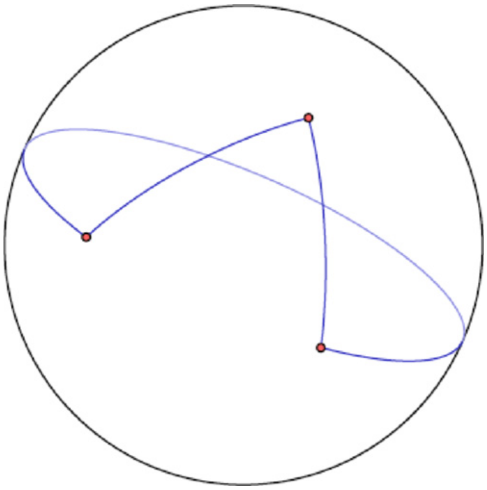

Fig. 3. Wade's figure.

Shortly thereafter the teacher had written the two conjectures on the board: $180 < s < 270$ and $180 < s < 360$ and asked, "Does somebody have an example of a triangle that might stretch the boundaries?" Adam and Pete suggested starting with a 90–90–90 triangle and then "shift those circles so they're not perpendicular, and you've got a triangle with more than 270." Adam, however, expressed concern that "the right angles at the bottom are getting smaller," which we note would be the case on the plane if the top angle were enlarged.

The transformation of a 90–90–90 triangle creates new triangles for discussion as data to support the claim that the sum is between 180 and 360. This transformational reasoning (Simon, 1996) is beginning to take the form of General activity in the sense that Pete is creating spherical triangles from other spherical triangles and not from planar triangle examples or properties. The distinction between Referential activity and General activity centers on the sources of students' organizing activity. In Referential activity the sources of comparison are the definition and properties of triangle on the plane. In General activity the sources of comparison are examples, properties and definitions that are new to students as part of their budding new mathematical reality of triangles on the surface of the sphere. That is, the starting point for creating new spherical triangles is a "weird" spherical triangle, indicating that students are simultaneously working with and constituting the new reality of spherical triangles. However, the argument is Referential in the sense that students are noticing the strong contrast between having a range of sums on the sphere instead of a singular sum of 180 degrees on the plane.

As the discussion continued the teacher asked another group to show their example, one that the other groups did not consider. The triangle that is discussed is depicted in Fig. 3.

Wade: Well, this is a case with three line segments on a [sphere] that do *not* form a triangle.
Mattie: Well, we don't think they do—[murmurs of disagreement from Mattie and Wade's fellow group members about whether their example is or is not a triangle].
Wade: First of all, use the first one as the equator, and you come around and you stop on the opposite side. So it goes completely around, like 300 degrees or something like that. Another line segment on the great circle, and you have a third line segment on the great circle, and they all intersect with each other only once, only once, only once [pointing to each of the line segments in turn].

Note that Wade's "only once" comments referred to the definition of triangle on the board. His construction was in reference to the planar triangle definition.

Mattie: You're saying two figures. Which one? Are either one of them a triangle or are they both a triangle?
Wade: They're both triangles.
Betty: But it doesn't finish the great circle [pointing to the definition on the board]. A triangle on a sphere is suppose to be three great circle segments.
Students: That's what he has. He has three great circle *segments*.

Wade and Betty were engaged in Referential activity as they clarified what was meant by the stated definition. In continuing this discussion, Greg makes explicit the Referential issue that the students were grappling with.

[Students murmur about whether Wade's drawing is or is not a triangle.]

Greg: Don't be prejudiced to what you think a triangle looks like on a plane. We have a different definition and a different universe. Don't be prejudice to what you think a triangle has to be, like this [illustrates a prototypical planar triangle with his fingers].

Greg's comments emphasize the importance of using the definition in the new setting and not using prototypes or properties from the plane to determine what will be considered a triangle. His comments also suggest that he recognizes this as a new mathematical reality that is different from the planar reality.

As the conversation continued one student asked about the angles in Wade's triangles.

Wade: We've got two 90s, like 90 and 90 and 300 or so. Well no, this is—This angle here would be a 270 and this would be 270 and this would be something like 60.
Greg: We've got way over 360.

After a brief tangent (on negative angles and non-closed figures) the class clarified that there are two different triangles involved with the sum of the interior angles of 480 and 600. In the discussion about these sums that followed, some students, clarified by Cassidy's eventual description, realized that triangles on the sphere come in pairs and that the sum of the interior angles of the paired triangles is 1080 degrees.

Cassidy: Well, my idea was that it takes three points to make a triangle and on a sphere, for each point there's 360 degrees on each point, so that's 1080 total degrees for all three points on a sphere. And then if you take the two triangles you drew on the board up there [Wade's example] and add the two angles and total all angles together, you get 1080.

This discussion is General activity in the sense that students were reasoning based on a property unique to triangles on the sphere. However there is still a Referential aspect to this activity in its attempt to create a constant angle measure in the way that 180 degrees is the constant for the sum of the interior angles of a planar triangle. In particular Adam's comments, both before and after Cassidy's quote above, have a strong Referential nature. They include the following statements. Note: in any subsequent transcript four ellipses [. . . .] denote deleted transcript.

Adam: Right, but is there a relationship between the sum of angles in the triangle?
[. . . .]
Adam: So, can we define a common property of a triangle? Something that's consistent with a triangle.
Teacher: You mean, like the sum of the angles is 180, something, like that sort of idea?
Adam: I agree, that kind of idea, because to me it's still kind of hard to—Granted that meets our definition, but the question we have to ask is our definition complete, to verify that that is truly a triangle on a sphere. Because that's a definition for a plane, a triangle on a plane.

This comment is almost a return to Situational activity where the class debated what the definition should contain. The sum of the angles being 180 degrees is a property that is such an integral part of students' concept image of triangle that it seems to be an implicit, if not explicit, part of the "definition" on the plane. In reference to this property on the plane, Adam indicated a search for a similar property on the sphere.

Adam: It [Cassidy's conjecture] should always work, but the point is, does it lead us anywhere?
Teacher: I don't know. Is it a useful property? Is the sum of the angles being 180 on a triangle on a plane a useful property?
Adam: Well, we know that they, basically they can't exceed it and still be a triangle.
Teacher: Oh, you're trying to use it sort of as a definition. Yeah?

[Adam agrees and Mattie wants to ask a question in the little time remaining in class.]

Mattie: Now the question is when we draw one of these traditional triangles, have we drawn, have we formed two triangles? We always have an inside and an outside.
Teacher: So, if you had a triangle kind of like this [draws three small segments connected, looking similar to a prototypical planar triangle], does that mean that we now have two triangles?
Students: Yes.
Georgia: Oh my gosh.
Adam: That outside thing is a triangle?!? That's what we're saying?

As class ended, the teacher indicated that Mattie's triangle did fit the definition. Mattie's reasoning was General activity in the sense that she was using a property of triangles on the sphere (that they come in pairs) to create an example of a new type of triangle on the sphere. However, Mattie's reasoning was also Referential in the sense that she was pointing out a property on the sphere that is different than on the plane, i.e., that the exterior angles of a triangle also make a triangle. In addition, Georgia and Adam's strong reactions to Mattie's creation were Referential in that they implicitly expressed how surprising this triangle is compared to their experiences with planar triangles.

4.2.3. Summary

In this section we discussed the Referential activity that students engaged in during the small group work and whole-class discussion of the second half of Day 6. This was defining activity in that the focus remained on creating what it would mean to have triangles on the surface of the sphere. It was Referential activity in that students used the slightly modified planar triangle definition along with the properties they associate with this definition to create and negotiate triangles on the sphere. The focus was on using a concept definition and creating a concept image of spherical triangle by generating examples and examining the properties of these examples, particularly what the sum of the interior angles of a spherical triangle could be. At this stage the students were using the definition-of their Situational activity, in particular their definition-of planar triangle as modified for the sphere, to determine whether the new examples were or were not examples of triangles on the sphere. This occurred both constructively, in terms of using planar similarities to correctly create and negotiate triangles on the sphere, and also in surprised or conflicted reaction to figures on the sphere that seemed to fit the definition of triangle but were quite different from students' concept image of triangle derived from the plane.

A journal entry written by Fred, a student in the group with Adam and Greg, summarizes some of this excitement and conflict.

> From the figure [Wade's group had] drawn, it didn't seem like it was a figure at all, but in close observation it was a triangle! Yes, a triangle. It was a triangle based on the definition we chose in class. The definition of a triangle matched up with the figure. Though this was true, the figure did not look like a triangle. I did not see the triangle until someone brought up that it was a triangle by definition. Better yet, there were two triangles! Yes, the inside AND the outside were both triangles. Another surprising observation was when Group 1 gathered information about triangles on a sphere and concluded that the maximum number of degrees that a triangle can have with respect to its angles is 1080 degrees [sic]. These observations [have] changed my view on spheres. All along I was thinking and limited to a 2-dimensional perspective.

Fred's journal entry highlights the Referential nature of students' activity both with references to a concept image of triangle from the plane and with applying the planar definition to the sphere. The focus of the excitement then becomes on using the planar definition to create the new concept image—that is, the new mathematical reality—of triangles on the sphere.

4.3. *General activity: creating concept images and concept definitions*

General activity is mathematical work within a newly constituted reality in which interpretations and solutions are independent of the original task setting. In our case, students' organizing activity with refined definitions and concept images of spherical triangles functions as models-for (or definitions-for) enlarging the new mathematical reality of spherical triangles in ways that do not refer to the plane. As such, students are creating and enlarging their concept image of spherical triangles and creating new concept definitions for specific classes of spherical triangles (see Table 1). The increasing independence from planar triangle imagery, with associated activities of enlarging, generalizing, refining, and structuring this new world of spherical triangles, is significant because it evidences the emergence and ongoing constitution of a new mathematical reality.

We begin the analysis of General activity by revisiting an excerpt from the previous section on Referential activity, and argue that students' Referential activity is, at times, intertwined and inseparable from forms of General activity. We pick up the whole-class discussion on Day 6 just after the teacher had written two students' conjectures about the sum of the interior angles on the board: $180 < s < 270$ and $180 < s < 360$. She then asked if anyone had an example that might "stretch the boundaries."

Adam: If you take—We did the perpendicular circle—
Teacher: Like on page 55 [triangle with three 90 degree angles]?
Adam: Right, that's a perpendicular. All you have to do is shift those circles so they're not perpendicular, and you've got a triangle with more than 270.
Teacher: You're saying, extend these lines a little bit [pointing to the top two segments of her picture of a 90–90–90 triangle on the board]?
Pete: Yeah, continue the rotation around [The teacher indicates a motion that increases the angle at the top vertex by rotating the top two sides outward.]
[. . ..]
Adam: Oh, but the right angles at the bottom are getting smaller.
Pete: No, they stay the same.
Betty: As long as the third line stays the same, they're perpendicular.
Adam: Oh, that's true. They're still perpendicular.

In the previous section we described the Referential aspects of the discussion leading into these arguments including students such as Amy and Adam referring to fixed sums for the interior angles of a planar triangle and the students working from a notion of triangle as the finite area enclosed by three lines (causing students to restrict their triangles on the sphere initially to those enclosed in a hemisphere). Adam's suggestion that increasing one of the angles in a triangle would decrease one or both of the other angles could also be construed as Referential to the extent that it was based on Adam's concept image of what would happen to a planar triangle.

We also want to emphasize that this excerpt is the beginning of General defining activity. Adam and Pete increased the interior angle at one vertex of the 90–90–90 triangle while maintaining the other two angles at 90 degrees to create new triangles for discussion and as data to support their statement that the sum is between 180 and 360. This transformational reasoning is the beginning of General activity in the sense that they were creating spherical triangles from other spherical triangles and not from planar triangle examples or properties. Moreover, it is an argument for a generalized conjecture on the sphere regarding the range for the sum of the interior angles.

We continue our discussion of General defining activity by next drawing on classroom excerpts that occurred on Days 7 and 8 in which students investigated the spherical triangle congruence question of Side-Angle-Side (SAS). That is, the question of whether two spherical triangles are congruent[3] if two sides and the included angle of one of the triangles are congruent to two sides and the included angle of the other triangle. On the plane, textbooks treat SAS in different ways, including as an axiom, as a definition of congruency of triangles, or as a theorem to be proved (Henderson, 1996). On the sphere, the SAS congruence statement is not true for all triangles, but it is true for a suitably defined class of spherical

[3] The students in this course had discussed several different definitions of angle congruence including two angles being congruent if they can be made to cover each other exactly (except for side length). The hint from the textbook for the SAS theorem on the plane strongly pushes students toward using a triangle congruence definition that two triangles are congruent if they can be made to lie on top of each other through a series of isometries.

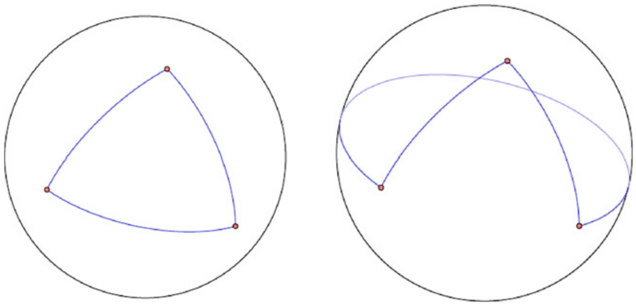

Fig. 4. A counterexample to SAS.

triangles. A primary component of the students' task was to decide if SAS is true for all triangles on the sphere[4]. If so, then they were to develop an argument to support this conclusion. If not, then they were to create a counterexample and define a suitable subset of all spherical triangles called "small triangles" for which SAS is true and develop an argument to support their claim.

We trace students' mathematical work on SAS, highlighting the back group's work. The mathematical work of this small group highlights many of the same issues that are brought forth in whole-class discussions, and hence we omit elaboration of the whole-class discussion.

4.3.1. Small group discussion Day 7

What was striking in students' initial response to the SAS task on the sphere was the immediacy with which some of them created counterexamples, evidencing a shift from models-of (or definitions-of) to models-for (or definitions-for) and the constitution of a new mathematical reality. To clarify, the relative ease with which counterexamples were created indicates that the surface of the sphere was now, for them, populated with a wide range of spherical triangles, however "weird" they once seemed. For example, Amy and Sam readily created a counterexample (see Fig. 4) and moved quickly into creating a definition for "small triangle" for which they thought SAS would be true.

Amy: So we've got side-angle-side. And one side would be going around there and the other one would be all the way around back like that.
Sam: But they're not congruent. So, it doesn't work. You have to limit it and say that it only works for small triangles. That you can't go all the way around the sphere.
Amy: And did we define small triangles?
Sam: No but we probably need to.
Amy: Yeah. [pause] Small triangles is an area less than half the area of the sphere?

At this point one might expect that Amy and Sam would next discuss whether Amy's posited definition for small triangle "works." That is, will Amy's definition eliminate the possibility of there being counterexamples to SAS? If not, how might her small triangle definition be revised? As it happened, however, the viability of Amy's tentative definition of small triangle did not get discussed because Tom and Cindy requested clarification as to why SAS does not work on the sphere. In the process of explaining the counterexample idea to them, Amy pointed to a fundamental difference between the plane and sphere, namely that on the sphere two points can be connected with two geodesic segments.

Tom: Why don't you think it will work on the sphere?
Amy: Because side-angle-side. There's one triangle where the line is there and there's another one where this line wouldn't be there and it goes all the way around the back. So they both have side-angle-side in common but one is this and one is. Do you follow? And they're not congruent but they have side-angle-side in common.
Cindy: If this is your angle, right? Not this one.
Tom: Oh. I see. Okay. Yeah.

Shortly thereafter Jay reiterated and expanded on what Amy said about the number of geodesic segments that can connect two points. In particular, he explicitly pointed out how, if the two points are antipodal, then there will be a "bunch" of geodesic segments connecting the points.

As this excerpt suggests, students' focus on the relationship between spherical triangles resulted in their creating concept images regarding the number of geodesic segments that connect two points, a fundamental property of the sphere. Creating such concept images was important because one, it provided an opportunity for Cindy and Tom to revisit and recreate the initially "weird" spherical triangles, and two, it afforded them an alternative way to create a small triangle definition. Recall that earlier Amy suggested that a small triangle is a triangle whose area is less than a hemisphere. In comparison, Sam

[4] The key property of the plane that makes SAS true is that two distinct points can only be connected with one straight line (geodesic) segment. On the sphere however, two distinct points can be connected with two different geodesic segments and thus there exists two non-congruent spherical triangles with SAS congruent. If the two points are antipodal, then there are an infinite number of geodesic segments that connect the two points.

suggested that, in light of *why* SAS fails to be true in general on the sphere, they define a small triangle in terms of the path one takes to connect vertices.

Sam: As long as you defined a straight line to be the shortest distance, not just any distance from B to C. That's why it falls apart in the sphere case because you can go outward from B and come inward on C. It could be the shortest distance. I mean maybe that's the definition of a small triangle is if you have points A, B, and C they're connected by a straight, shortest distance line.

Amy: Okay. Yeah. I like that too. I like that better. It's more concrete.

Amy likely saw this small triangle definition as more "concrete" because it is directly tied to how one constructs triangles (and counterexamples to SAS). In comparison, Amy's initial idea for defining small triangle in terms of area is a property of the object that would be derived from the triangle after it is constructed, rather than being central to the construction process.

Spherical triangles had become familiar to these students and available for use, indicating that they were working within a newly constituted mathematical reality. Within this new reality students began to create concept images for underlying properties of the sphere and to create and negotiate a definition of small triangle to suit their goal of making SAS true. Such mathematical work is indicative of General defining activity because of its increasing independence from planar imagery and its use of prior mathematical activity as models-for structuring the new world of spherical triangles.

Students continued to work on the SAS problem at home, and discussion on this problem continued the following class session, first in small group and then in whole class. We continue to examine the back group's work in order to further illustrate General defining activity.

4.3.2. Small group discussion Day 8

Cindy began the conversation about her work on SAS by telling the group her definition of small triangle, to which Jay voiced some concerns.

Cindy: The triangle covers less than half of the sphere, which can be done when there are more than two triangles.

Jay: I would just say the, the triangle covers less than half the sphere.

Cindy: Yeah, but my definition is, my definition is that it doesn't have to be small, it means something that can be congruent to something else on a sphere.

Jay: But you still have a triangle, even if there is only one on the sphere. . .One thing that I don't like about yours is that you have it defined as something that can be congruent to another one. . .You're wanting to know if you can apply SAS to say two triangles, and you don't know if you can apply that unless you know the triangle's congruent to another. It's like a circular thing.

What is important to highlight in this excerpt is that Jay and Cindy negotiated, implicitly, criteria for what makes a good or useful definition. We note that students engaged in similar negotiations in Situational activity, and we see here again such negotiations. What makes negotiating criteria for a good or useful definition fall within the realm of Situational or General defining activity are the context and history of the students. In this case, the negotiation of criteria for a definition is done in the context of refining their new reality of the spherical triangles and therefore falls with the realm of General defining activity.

As the small group discussion continued, Jay put forth a different definition for small triangle, namely that "the sum of the [interior] angles is less than 540 degrees." In a rather informal way, he conjectured that "anything [any small triangle] that covers less than half a sphere, the sum of the angles is going to be less than 540 degrees." This is one direction of what one would have to prove to show that the two definitions are equivalent. Several other students made similar informal comparisons between different definitions. Such informal conjectures about the equivalence of definitions are another form of General defining activity because this work is creating concept images that structure the new reality of spherical triangles.

Further General defining activity unfolds from Jay's proposed definition of small triangle. In particular, Jay's definition sparked Amy to clarify and enlarge her concept image of spherical triangles. Amy said, "I'm just trying to think about what's the largest triangle that you could fit on a half, a hemisphere? And how could you possibly measure these angles, and how it actually helps you." Jay, Amy and the rest of the group then explored how one can start with a quartersphere and "push" out the sides to get a small triangle that almost fills up the hemisphere. This type of creating activity is reminiscent of the whole-class discussion from Day 6 discussed at the beginning of this section. As we argued before, this type of activity falls within the realm of General activity because rather than using the plane as the frame of reference for creating new spherical triangles, students are using their newly constituted world of spherical objects to create new triangles and new concept images for spherical triangles.

Another interesting result of this conversation is that it motivated Amy to create a new small triangle definition, one that was entirely different from what she had developed the day before. The definition that she came to class with was that "each straight line segment in a small triangle takes the shortest path between each set of points." Her new definition, inspired by her group's discussion, was "that each angle in the small triangle is less than 180 degrees." We again see this as a form of General activity because Amy took an existing definition, one offered by a member of her small group, and used it as the inspiration to develop a different small triangle definition.

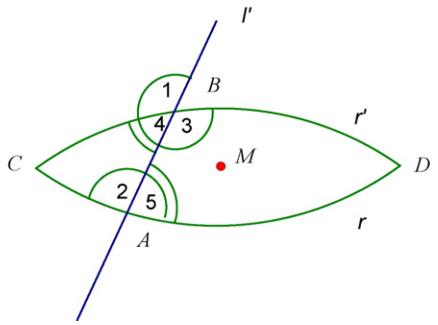

Fig. 5. Russ's illustration.

4.3.3. Summary

In this section our analysis characterized students' work as General activity since this work resulted in enlarging, generalizing, refining, and structuring the new world of spherical triangles. These activities were commensurate with creating concept images and concept definitions. A paradigmatic example of general activity was evidenced by students' ability to define a subset of spherical triangles to suit their goal of satisfying a congruence statement, such as SAS. Defining a subset of spherical triangles was a way to structure their new reality and create new concept images for the objects in this new reality. Creating a small triangle definition necessitated a focus on the *relationships* between these new mathematical objects (spherical triangles). This is different from Referential activity that, in part, focused on what is true about a given object in relation to the more familiar planar triangles (e.g., Is it possible to have a triangle with two obtuse angles? Can the sum of the interior angles be more than 180 degrees?). Finally, defining a class of spherical triangles for which a congruence statement, such as SAS is true, required positing a tentative restriction to the world of spherical triangles, examining whether the posited restriction/definition does the job intended, and refining or revising the definition if it does not. In this sense, defining and proving may be closely linked[5].

4.4. Formal activity: using established concept images and concept definitions

Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for (or definitions-for) activity. In our context, students reason about spherical triangles in ways that reflect new structural relationships between objects in the new mathematical reality and consequently use definitions as links in chains of reasoning without having to revisit or unpack the meaning of these definitions. At the Formal level, student activity shifts from a major emphasis on *creating* what a triangle is to a primary emphasis on *using* concept definitions and concept images of triangle that have been previously established.

In many cases of Formal activity, the definition and properties of spherical triangle remain completely in the background or are referenced, but indirectly. There are also instances of Formal activity where definitions and concept images are used more explicitly but defining is Formal in the sense that the concept images and triangle definitions are not questioned, nor revisited, and the new mathematics for students is not focused on spherical triangles per se, but rather on some new content. Below we will describe an example of the former briefly and then go into more detail on an example where definitions and concept images are used more explicitly.

The following examples are drawn from student discussions on Day 22, the last day of class before the final exam, when students revisited a problem that they had attempted on Day 14. In setting up the problem at the beginning of Day 22 the teacher stated a variation of the problem that some students had worked with previously on Day 14 and in their homework.

Teacher: Some people want to recast this slightly. Instead of talking about the line *l* and its midpoint, they wanted to just recast it as the midpoint of the lune itself. So, if there's a line that does not pass through the midpoint of the lune, could it cut at congruent angles?

4.4.1. Whole-class discussion

After a period of about 25 min of small group work, Russ from the front group presented his group's proof that a transversal that does not pass through the midpoint of the lune will not intersect the two sides of the lune at congruent angles. However, after some discussion, the class determined that his proof hinged on the construction of the transversal such that both of its intersections with the lune occur on the same side of an equator line through M (see Fig. 5). His proof did not address the more general case of the transversal having one intersection on each side of the equator of the lune.

Formal activity is recognizable in two parts of Russ's presentation and the whole-class discussion of his proof. First, there was an implicit use of the definition or properties of a triangle on a sphere simply in his claim, at one point in the

[5] See Larsen and Zandieh (2005) for further description of this phenomenon.

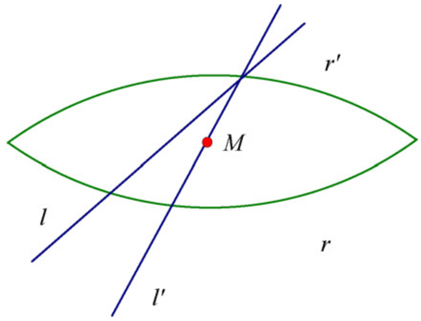

Fig. 6. Cindy's illustration.

proof, that the construction creates two triangles, "So, now we have triangle ABC is congruent to triangle BAD by angle-side-angle." Note that he did not explicitly state that there are two triangles in his drawing until he made the claim for their congruence. His claim that the two figures are triangles had no stated warrant and was in the context of making a point about something else. In this example, not only was the focus of the proof not about triangle, but additionally the discussion of any definition or properties of triangle was completely in the background. However, it is Formal activity on a sphere because of the implicit use of a definition or properties of a triangle necessary to claim creation of triangles in this figure.

The second occurrence of Formal activity in Russ's discussion was his indirect use of the definition of two triangles being congruent. After claiming that the triangles are congruent he stated, "So that means that CA, side CA, this one down here, is congruent to BD by congruent parts." One student asked why knowing that the two triangles are congruent allows us to claim that CA is congruent to BD. Several students said immediately, "Corresponding parts." This student's concern prompted the teacher to bring the discussion of triangle congruence to the foreground.

Teacher: Okay, so on the one hand we have some congruence theorems that tell us if we have certain of the parts, we can prove congruence, right? Now, when we say we're proving that they're congruent, what do we mean by that?
Russ: Corresponding parts are all equal.
Teacher: I mean, one way to say what it means that two things are congruent is that every single part matches up. So really, it's more *by definition* of being congruent, all the corresponding parts have to match up [teacher's emphasis].

Here the activity for Russ and some other students in the class is Formal with respect to the definition of triangle congruence. These students used the definition, in the somewhat implicit form of "corresponding parts," as a warrant in their deductive argument without focusing on the meaning of triangle congruence. The student question, however, turned the activity of the whole class briefly toward defining in the sense of articulating and clarifying a definition and hence created a clarification for the mathematical reality of triangles on the sphere.

In the whole-class discussion, no one asked about the need to check whether the two triangles are small triangles, which allows the use of ASA on the sphere. This concern came up in a small group discussion later in the class and is our final example of Formal activity. The example starts after the teacher sent the students back to small groups to work on the case in which the intersections of the transversal and lune are not on the same side of the equator.

4.4.2. Small group discussion

About 10 min into the small group discussion, the teacher visited the back group and discussed with them a figure that Cindy had drawn related to extending Russ's proof to the case where the transversal had intersections on either side of where an equator would be located. Cindy also had drawn a second transversal that passes through both the midpoint of the lune and one intersection of the first transversal (see Fig. 6). The teacher was not sure whether Cindy's sketch would lead to a proof but encouraged the group to further consider it by summarizing Cindy's construction and asking whether the construction was always possible.

Cindy was sure that the construction was always possible. As the teacher left, the students began to discuss how this construction may give a proof by contradiction.

Jay: With this you can prove that this whole triangle is congruent to that whole triangle [referring to the triangles on either side of *l*].
Cindy: I already did.
Jay: And then you can prove that this whole triangle is congruent to this whole triangle [referring to the triangles on either side of *l*′].
Cindy: And therefore, you have a contradiction.
Jay: Why though?
Cindy: Because one includes this [small middle triangle] and the other doesn't. This is congruent to all of this.
Jay: OK, I think the contradiction is that this triangle and this triangle [on either side of *l*], together they add up to the whole lune, and this triangle and this triangle [on either side of *l*′] add up to the whole lune. Yet, this [triangle to the left of *l*] is smaller than that [triangle to the left of *l*′]. So, yeah. Works for me.

Cindy seemed to see that you have a congruence with one triangle and then another congruence with a triangle that is a subset of the first triangle. Jay elaborated this further after being pressed for further explanation by Amy and Cindy.

Jay: Okay, so the triangle with l' as a base over here is congruent to the triangle with l' as a base over here. Okay, you can get that from angle-side-angle. Also, assuming l is a parallel transport, you can get the triangle with l as a base.

Cindy: But they intersect. Can we do something about intersecting because they're intersecting? If they're parallel transports they should intersect?

Jay: They're not parallel transports of each other. Okay so then we get that this triangle with l is equal to this triangle with l that way. Well, they both—all four triangles equal half the lune. So then this l, this triangle though is clearly less than this triangle. So, there's your contradiction. It is somewhat by picture, but it's the way we constructed it. The contradiction is you've got all four triangles' area is equal to half the area of the lune and yet this triangle has clearly got less area than this one. So, there's your contradiction.

Note that in the above, any discussion of triangle definitions was still very much in the background. However, as the students started to write up their proof in a more finalized version, more detailed questions emerged that brought the small triangle definition to the foreground.

Sam: Does ASA only work for small triangles or very small triangles?

Jay: Small triangles. The very small triangles was the EEAT [Euclid's Exterior Angle Theorem] thing.

Amy: Are these triangles always going to be small? I don't know if that's true. The proof that I've constructed in my head that I'm happy at least at some level—happiest with, involves ASA. And if, is that, I think that's limiting us.

In the conversation that follows the students considered a lune with angle greater than 180 degrees and whether or not the triangles formed would be still be small triangles. Jay summarized this conversation as follows:

Jay: Yeah, okay but a lune will be antipodal but when you divide it, this should be able to fit in an open hemisphere, and this should be able to fit in an open hemisphere because you're only taking half of the lune. I mean, you're pushing it right to the limit because you could have 359 and 60 min, in degrees.

Tom: I see what you're saying.

Jay: But that is a good point. ASA only works for small triangles.

In the above discussion the students recognized that one step in their proof needed further justification. The ASA triangle congruence theorem only holds true on the sphere for small triangles. This discussion brought the definition of small triangle to the foreground of the discussion in the sense that the students had to argue whether or not their construction would create only small triangles. Jay used the class's primary definition of small triangle, that of a triangle that fits in an open hemisphere, to make his argument. He imagined pushing the lune angle "right to the limit" and explained that when this is divided in half by the parallel transporting line, then the two triangles created would be less than half a hemisphere.

Thus the definition of small triangle was discussed explicitly in terms of making sure that the conditions necessary for the ASA theorem were satisfied. This is a case of Formal activity in the sense that it illustrates the explicit use of a definition in a chain of reasoning used to prove a theorem about something very different than the concept involved in the particular definition. For example, here we see a definition of small triangle used in the service of making this proof about parallel transported lines more rigorous. No revisions or reflections on the definition itself were required.

4.4.3. Summary

In the three previous sections students engaged in defining activity regarding triangles, defining planar triangles, and creating and exploring spherical triangles. In this final section, student activity is no longer focused on triangles. Triangle definitions and concept images are both used as is and in the service of other goals (see Table 1).

These examples are all types of Formal activity at various levels of explicitness. However, they are all Formal with respect to triangles on the sphere in that the definitions or concept images of triangles on the sphere, whether in the foreground or background, are used in the service of some other purpose. In the example tendered, the purpose was the exploration and proof of properties of parallel transported lines on the sphere. None of the definitions or properties of the triangles are reconsidered, recreated nor explored. Rather, they are restated for clarification purposes or discussed in terms of whether they can be used in the given situation.

5. Discussion

In the previous section we traced a trajectory from informal or less formal activity to more formal activity following the four levels of activity described by Gravemeijer (1999) in the emergent model perspective. We next step back from the details of this trajectory and discuss how the four levels of activity relate to using and creating concept images and definitions. From there we bring out further comparisons to Gravemeijer's (1999) work on emergent models and conclude with a reflection on implications for instructional design.

While the literature often points out discontinuities between concept image and concept definition, our analysis describes a trajectory from less formal to more formal that allows students to make greater and richer connections between their concept image and concept definition. We also note that although the mechanism of our trajectory is a series of definitions that the students create, use and contrast with their concept images, the ultimate purpose of the trajectory is the creation, not of definitions, but of new conceptual understanding, in this case the new mathematical reality (for these students) of triangles on the surface of the sphere.

Our analysis incorporates the themes in the literature of creating and using concept images and concept definitions and makes comparisons to the four levels of activity that characterize the heuristic of emergent models. As such, we highlight the following four forms of definitional activity: creating a concept definition, using a concept definition, creating a concept image, and using a concept image. Table 1 shows that, in broad terms, each of the four Emergent Model levels of activity emphasizes two of the forms of definitional activity while the other two forms are used more sparingly.

In Situational activity the students were engaged in defining in that they were creating a planar triangle definition based on using their rich concept image of planar triangle. In other words, they were using their concept image for creating a definition. The extent to which they were using a definition or creating a concept image was much less. However, creating a definition also involved negotiating and refining definitions and this in turn involved using the definitions. For example, students interpreted (used) a definition by finding examples that fit a proposed triangle definition that may or may not agree with their concept image of planar triangle. In this case the interpretation (use) of the definition served the purpose of furthering the negotiation and refinement (creation) of the definition. In addition there was a small sense in which the students were creating their concept image of planar triangle. To formulate a definition, students were organizing their knowledge about planar triangles and examining potentially new properties or examples of planar triangles. For example, students considered whether they wanted to call something that looks like a single line segment or a single point a triangle. Thus, although our students' Situational activity primarily was focused on creating a definition by using their concept image, there was some sense of using definitions and creating conceptual understanding as well.

Students' Referential and General activity focused much more on creating a concept image, in this case for spherical triangles, and hence Referential and General were the primary levels in terms of the creation of the new mathematical reality of triangles on the sphere. In Referential activity, the focus was on interpreting (using) a planar triangle definition in the new context of the surface of the sphere. Thus students were involved in a variation of the definition game in that they were asked to interpret a familiar definition in an unfamiliar setting. They used this specified definition of triangle to begin to create a concept image of triangles on the sphere. In other words, they were creating a concept image by using a concept definition.

Although the focus was on using a given concept definition for creating a concept image, students did a bit of definition creation in that they had to reword their planar triangle definition by reinterpreting the notion of straight line segments as geodesic segments on the sphere. The students were also engaged in using their concept image of a triangle on a plane in that they contrasted it with the new triangles that they were creating on the surface of the sphere. Students focused on creating examples of spherical triangles and exploring possible characteristics of those new triangles, but they did this in reference to (i.e., by using) their knowledge of the concept definition and concept image of planar triangle. Thus the students' Referential activity focused on using a concept definition (and to a lesser extent a concept image) for creating a concept image.

In General activity, the students continued to work on creating the new mathematical reality of triangles on the sphere. The focus of this activity was two-fold. First, students generalized from the examples of spherical triangles they had created and the properties of these triangles noted in Referential activity. Second, students created new definitions, such as for small triangle, that helped delineate new theorems for spherical triangles. Students thus were creating a concept image for triangles on the sphere, and they were creating definitions that were part of this new mathematical reality. Accomplishing this, however, included the students' using their developing concept images of spherical triangles and also using their definitions for spherical triangle and small triangle. The primary difference between General and Referential activity lies in the source of one's concept images. In General activity the primary concept images and concept definitions come from the realm of spherical triangles. In comparison, concept images in Referential activity come from the realm of planar triangles.

In the fourth level of activity, Formal activity, the new mathematical reality of spherical triangles was largely established for these students. The students thus focused on using concept definitions and images from this reality in the service of other goals. For example, they used definitions of triangle congruence and of small triangles on the sphere as part of their justification for steps in proofs that were not directly about triangles. They also used their concept images of spherical triangle in constructing triangles within lunes and in making conjectures about the relationships between parts of these triangles that led to theorems and proofs about lunes and parallel transported lines. Thus Formal activity focused on using concept definitions and concept images related to spherical triangles without unpacking or reflecting on the content of those definitions or images. Creation of a concept image for spherical triangle only occurred in the limited sense of students developing more examples of how and in what cases spherical triangle theorems and definitions could be applied (e.g., whether the ASA theorem could be applied in a given situation). Students did not create any new definitions related to spherical triangles but they did recall and (for some students) refine their definitions of triangle congruence or small triangle on the sphere.

In summary we saw in our data that each of the four levels of activity put a greater emphasis on two of the four creating or using constructs. The emphasized notions were (also see Table 1):

Situational activity: using a concept image, creating a concept definition.
Referential activity: using a concept definition, creating a concept image.
General activity: creating a concept definition, creating a concept image.
Formal activity: using a concept definition, using a concept image.

In addition, we do not intend to imply that the four levels of activity must always have precisely these pairs of relationships. Referential and General activity will always be the focus of the creation of a new mathematical reality and hence creating a concept image will likely always occur strongly in these two levels. Similarly, Situational activity and Formal activity are more likely to focus on using a concept image or a concept definition for which the student is already familiar. In Formal activity this means using concepts and definitions developed through the four levels of activity. These relationships are highlighted in each section but do not preclude bits of the other pairs occurring in each level of activity.

In addition, recall that the concept being referred to in the phrases "concept image" or "concept definition" above changes as students progress from activity involving the familiar planar triangles to activity with spherical triangles and specific subsets of spherical triangles. As such, the term "model" does not refer to a single inscription or concrete object. As stated earlier, models are student-generated ways of organizing their activity with mental or physical tools. We describe a series of definitions that are related in that each definition emerges from activity with the previous definition. The series of definitions includes planar triangle, spherical triangle, triangle congruence, and small triangle.

To conclude, we comment on a practical implication of our analysis that enriches the instructional design theory of Realistic Mathematics Education by extending the emergent model heuristic to the case of defining. Characterizing defining in terms of four levels of activity (Situational, Referential, General, and Formal) together with elaboration of these levels as they relate to creating a concept image, using a concept definition, creating a concept image, and using a concept image (see Table 1) offers others a framework in which to develop sequences of learning activities for student learning in other content areas. Indeed, as Streefland (1993) and others (e.g., Gravemeijer, 1999; Yackel, Stephan, Rasmussen, & Underwood, 2003) demonstrated, each enactment of RME provides an opportunity to further develop the broader instructional design theory of RME.

For example, if one's goal is for students to develop and refine their understanding of function, then one can use the organization of activities in Table 1 as an outline for developing a problem task sequence. Using Table 1 as an organizational tool for planning instruction has the advantage that it is thoroughly grounded in the instructional design theory of RME and it points the designer to important themes in the literature that can inform the design. If one works from the point of view that definitions are an essential aspect of mathematical activity, then the analysis presented here offers a theoretically grounded vision of how such activity can unfold.

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