Contents

42. Design research within undergraduate mathematics education: An example from introductory linear algebra

Abstract 907
1. Introduction 907
2. Design and development stage 908
3. Theoretical framework 909
4. Research agenda and data collection 910
5. The need for design research 911
6. The magic carpet ride sequence 912
7. Conclusion 920

Key sources 921

References 922
Design research within undergraduate mathematics education: An example from introductory linear algebra

Megan Wawro, Chris Rasmussen, Michelle Zandieh & Christine Larson

Abstract
The aim of this chapter is to detail an example of design research in undergraduate mathematics. The results we present is an instructional sequence known as the Magic Carpet Ride (MCR) sequence and the underlying theoretical rationale for the sequence. The example, which fosters students' reinvention of span and linear in/dependence, comes from a sequence of classroom teaching experiments conducted within inquiry-oriented linear algebra classrooms in the Southwestern United States. Within this chapter, we emphasize how the Realistic Mathematics Education heuristic of emergent models and Gravemeijer's (1999) four levels of activity inform the MCR sequence.

1. Introduction
Over the past twenty years, mathematics education research has begun to shift away from an exclusively cognitive focus to one that acknowledges the situated nature of student activity within the classroom and broader communities of practice (e.g., Lave, 1988; Wenger, 1998; Yackel, Rasmussen, & King, 2000). A second shift in the field reflects the tight integration and equal status of theory and practice. The design research approach of classroom teaching experiments (Cobb, 2000), which is central to our work in linear algebra, is especially well suited to both of these shifts in the field.

As illustrated in Figure 1, design research in a classroom teaching experiment consists of a cyclical process of ongoing analysis of student reasoning and simultaneous task design and conjecture modification regarding the possible paths that students' learning might take (Gravemeijer, 1994; Cobb, 2000). The products of classroom teaching experiments often include both theoretical advances (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003) as well as empirically grounded instructional innovations (e.g., Larsen, 2009; Rasmussen & Keynes, 2003; Stephan & Akyuz, 2012). In other words, according to Treffers (1987), the purpose of design research can encompass developing, testing, and revising instructional sequences.
The results we present in this chapter are a case of the latter. That is, we present both an instructional sequence known as the Magic Carpet Ride (MCR) sequence, which was a product of design research within the context of an undergraduate mathematics classroom, and the underlying theoretical rationale for the sequence. The MCR sequence arose as a product of the following research question: What instructional sequence could be created and enacted to foster students’ guided reinvention of the concepts of span and linear independence, and how could the sequence leverage students’ intuitive knowledge towards the development of more formal ways of reasoning about these two concepts? The MCR sequence and the theoretical basis for it grew out of three semester long classroom teaching experiments at two institutions with three different instructors, each of whom were central members of the research and curriculum development team. In the subsequent section we detail more specifically the iterations of our design research.

2. Design and development stage

In general, our design research adheres to the principles illustrated in Figure 1. Our work in instructional design is informed by a theoretical framework (that of Realistic Mathematics Education, detailed in the subsequent Theoretical Framework section). The data collected during task implementation feed the ongoing analysis of the task and of student thinking, and this analysis is grounded in theories of learning and associated research results. The findings from this analysis feed back into the next iteration of the design cycle by informing any refinement or alterations of the task sequence and/or the implementation of the task. More detail about the structure of the specific iterations is given in Figure 2 and the remainder of this section.

![Figure 2: Schematic of the iterative cycle of task implementation and refinement](image)

During the first stage of developing a new task sequence, labeled Creating the Initial Task Sequence in Figure 2, we draw on several sources of knowledge and influence. First, we consider the learning goals we have for the students. What mathematical ideas do we want students to develop deep conceptual understanding of, and where do those ideas fit into the broader scope of the course as a whole? We then begin to think about what tasks we could
create that have the potential to facilitate those desired mathematical understandings. This task development is deeply influenced by our belief that tasks should have the potential to elicit students’ current/intuitive ways of thinking about mathematical ideas that then can be leveraged towards more formal mathematical reasoning about the concepts involved. The implications of this are captured by Gravemeijer (2004), who states: “For the instructional designer this implies a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing” (p. 106).

At this first iteration of the task sequence, at its creation stage, we cannot be certain how students will engage in it. Thus, we have to draw on our knowledge of student thinking, both in general and in specific. In general, we rely on research regarding how students learn mathematics. For example, Tall and Vinner (1981) introduce the notion of concept image and concept definition as lenses through which to analyze how individual students make sense of particular mathematical concepts. Another example would be to consider development through the lens of participating in mathematical activities such as symbolizing, algorithmatizing, and defining (Rasmussen, Zandieh, King, & Teppo; 2005; Zandieh & Rasmussen, 2010). Specific to linear algebra, we draw on our knowledge of student understanding of particular mathematical concepts from literature or past teaching experiences. For example, literature such as Harel (1997), Stewart and Thomas (2010), Trigueros and Possani (2011), and Wawro (2011) present research regarding students’ understanding of span and linear independence. Thus, incorporating these aspects, we write a first draft of the new task sequence.

Next, in general, we pilot the task sequence with a subset of students outside of the actual class (labeled Piloting the Task Sequence in Figure 2). This involves a researcher video recording a set of students trying the task, and he/she interacts with them as a teacher might. This data is then reviewed to gain information about how the students engaged with the task. What ways of thinking were elicited? What of these ways of thinking were productive, and which were possibly problematic? Did the task seem to have the potential to facilitate the development of more formal ways of reasoning about the concept(s) involved? We take the information gleaned from analyzing the task’s first implementation to inform any possible refinement of the task. This is labeled Analyzing and Refining the Task Sequence in Figure 2. We next use the refined task sequence in a classroom environment; this is labeled Teaching with the Task Sequence in Figure 2. This begins the second iteration of the design research cycle. This cycle of refining the task and implementing it (seen in the arrows between the upper two boxes in Figure 2) is continued until some balanced state is achieved; of course the task and its implementation are never completely stable, in that an inquiry-oriented classroom requires responsiveness and adaptation to student thinking.

3. Theoretical framework

In terms of our theoretical commitments regarding task design, we use as guiding heuristics the principles of Realistic Mathematics Education (RME) (Freudenthal, 1991; Gravemeijer, Bowers, & Stephan, 2003) to create tasks that afford students opportunities to progress along a continuum from informal to more formal ways of reasoning. In particular, our use of RME leverages the RME emergent model design heuristic, in which students first develop models-of their mathematical activity, which later become models-for more sophisticated mathematical reasoning (Gravemeijer, 1999). Models are defined as student generated ways of organizing their activity with observable and mental tools (Zandieh & Rasmussen, 2010). The shift from model-of to model-for, which is accompanied by the creation of a new mathematical reality or
set of objects, is further explicated with four levels of activity: situational, referential, general, and formal (Gravemeijer, 1999; Rasmussen & Blumenfeld, 2007; Zandieh & Rasmussen, 2010). In brief, situational activity involves students working toward mathematical goals in an experientially real setting. Referential activity involves models of that refer (implicitly or explicitly) to physical and mental activity in the original task setting. General activity involves models for that facilitate a focus on interpretations and solution processes independent of the original task setting. Finally, formal activity involves students reasoning in ways that are independent of the original setting and reflect the emergence of a new mathematical reality.

For example, in an undergraduate Euclidean and non-Euclidean geometry class, Zandieh and Rasmussen (2010) detail each of these four levels of activity to analyze students’ mathematical progress. In situational activity students used the previous experiences with triangles to create a planar triangle definition, largely based on using their rich concept image of planar triangle. In referential activity, students’ focus was on interpreting a planar triangle definition in the new context of the surface of the sphere. They created examples of spherical triangles and explored possible characteristics of those new triangles, largely in reference to their knowledge of the concept definition and concept image of planar triangle. In general activity, students continued to work on creating the new mathematical reality of triangles on the sphere. The focus of this activity was two-fold. First, students generalized from the examples of spherical triangles they had created and the properties of these triangles noted in referential activity. Second, students created new definitions, such as for small triangle, that helped delineate new theorems for spherical triangles. In formal activity, the new mathematical reality of spherical triangles was largely established for these students as they used definitions of triangle congruence and of small triangles on the sphere as part of their justification for steps in proofs that were not directly about triangles.

4. Research agenda and data collection

Our work is driven by the notion that teaching and research are tightly interrelated, in that one necessarily informs the other. This reciprocal relationship between theory and practice motivates us to investigate how students learn particular ideas in mathematics, as well as to seek out and develop theoretical tools that meet the pragmatic needs of teachers and researchers. The aspect of our research agenda most pertinent to this chapter regards the learning and teaching of linear algebra, specifically with respect to the content of span and linear independence. How do students reason about these concepts, and what curriculum may facilitate students’ learning? Other aspects of our research agenda, such as developing methodological tools for documenting the complex nature of mathematical development in the classroom (e.g., Rasmussen, Wawro, & Zandieh, 2012; Rasmussen, Zandieh, & Wawro, 2009), are beyond the scope of this chapter.

The data that we collect in conducting design research includes video recordings of each class session with at least two cameras, copies of student written work during class, copies of all assignments, and individual student interviews conducted at the beginning, middle, and end of the semester. The purpose of the individual interviews varies depending on our progress in developing instructional sequences that enable students to build on their current ways of reasoning. In particular, when we are in the early stages of design research we typically use beginning of the semester individual interviews to gain greater insight into students’ intuitive or informal ideas, as well as piloting instructional tasks that will be used in subsequent class sessions. When our instructional sequences are more mature, we typically use beginning of semester individual interviews to test conjectures about students’ intuitive or informal thinking.
End of the semester individual interviews, both early and later in our design research, typically assess the extent to which students actually developed in their understanding of the intended concepts.

The classroom video recordings of class sessions capture both small group work and whole class discussions, and this data is reviewed by the research team on a regular basis (at least once per week). This regular, ongoing analysis is structured by the following interests: How were instructional tasks interpreted and how did they function to promote students' mathematical thinking? Are students routinely explaining their thinking listening to others' explanations and challenging or questioning explanations when they disagree with what others' say? Are small groups functioning well in terms of enabling each student to mathematically contribute and to debate differing points of view? How are particular teacher actions working to create an environment that is conducive to individual and collective mathematical progress? To a large extent, these interests grow out of our theoretical commitments in which learning is viewed as both a social and psychological process (Cobb & Yackel, 1996). Retrospective analysis of the data sources also provides opportunity for theory building about the developing instructional sequence. Regarding the example in this chapter, we used Gravemeijer's (1999) four levels of activity as a starting point to lend form to the MCR, but the refinement of the MCR sequence and its framing within the four levels was made possible by the cyclical process of design research. This is our movement towards building a local instruction theory (Gravemeijer, 1999; Nickerson & Whitacre, 2010).

The data we present in this chapter comes from the third classroom teaching experiment in an introductory linear algebra course during the 2010 spring semester at a large southwestern public university. There were 30 students enrolled in the course, and most had completed three semesters of calculus (at least two semesters was required). Approximately half had also completed a discrete mathematics course, and 75% of students were in their second or third year of university. Most students in the class were majoring in Computer engineering, Computer science, Mathematics, or Statistics.

We begin the remainder of this chapter by describing how our approach of beginning the semester with systems of equations was problematic and thus led to a significant shift. We then detail the result of our shift: the creation of an innovative instructional sequence that supports students' reinvention of the ideas of span and linear independence/dependence. The instructional sequence described here began on the first day of a linear algebra course and facilitates the opportunity for students to intuitively explore the concepts of span and linear in/dependence and reinvent their formal definitions. The chapter concludes with an overview of the theoretical basis for the instructional sequence in terms of the four layers of activity that comprise the RME emergent model heuristic.

5. The need for design research

In our classroom teaching experiments, we work to develop tasks that afford students the opportunity to make significant progress in developing and/or reinventing fundamental mathematical ideas. In our early work in linear algebra, we began the semester by asking students to work on a real-world problem that they were likely to model with systems of linear equations. We chose this approach because of (a) the prevalence in linear algebra textbooks in the United States to begin with systems of equations and Gaussian elimination, and (b) its potential to draw on students' prior knowledge and experiences with systems of equations from coursework in high school or college algebra. Our intent was to use this real-world problem to
introduce alternative notations for modeling the situation, such as matrix equations and vector equations (involving linear combinations of vectors).

Analysis of student interviews conducted during our first classroom teaching experiment, however, revealed a lack of coordination between students’ symbolic representations and geometric interpretations of a matrix times a vector (Larson, 2010). For example, students’ geometric interpretations were underdeveloped with respect to connecting geometric conceptualization of linear combinations of vectors with matrix multiplication. In our view, this interpretation is particularly important for students in making sense of ideas relating to span, linear dependence and independence.

Our retrospective analysis about students’ difficulties points to the original task setting as problematic. It was difficult for students to interpret geometrically, and it did not generalize to situations with non-integer or negative solutions. Furthermore, the problem setting failed to generate a need for notations introduced (such as matrix equation and vector equation notation).

As we planned for subsequent iterations of the teaching experiment, we felt a need for a switch: to begin the semester with a geometrically motivated task that emphasized vectors as the mathematical objects of focus, rather than systems of equations. This would create a need for a geometric interpretation of linear combinations of vectors, which could then be leveraged for ideas relating to span and linear in/dependence. Furthermore, the need to develop more sophisticated solution techniques through systems of equations would emerge naturally from students’ work with span and linear in/dependence. We also viewed it as crucial that this context generalize nicely to situations with non-integer valued and negative coefficients. Finally, this alternative problem situation that focuses on vectors as the objects of inquiry and investigation could also be an intuitive starting point for students.

The Magic Carpet Ride sequence, thus, grew out of analysis of student thinking and a reconsideration of our learning goals for the students in the course. With respect to the design cycle explained in Figure 2, the MCR was first created during summer research team meetings. The research team developed the context of travel as a potentially viable one to leverage towards students’ development of the notions and vector equations, span, and linear independence. The team debated the development of the specific tasks (such as task wording, symbolism and numerical values to use, etc.). The sequence implementation and refinement cycle illustrated in Figure 2 occurred over three semesters at two institutions with three different instructors, each of whom were central members of the research and curriculum development team. During and after each implementation, the research team would analyze student thinking and the role of the teacher within the development of the classroom mathematics. This led to alterations with the sequence, such as changes in wording of the tasks or key questions that the teacher could ask. The subsequent summary of the MCR, comprises the remainder of this chapter, is the most recent version of the task sequence.

6. The Magic Carpet Ride sequence
The Magic Carpet Ride (MCR) instructional sequence is a set of four tasks that makes use of an experientially real problem setting to support the reinvention of the concepts of span and linear in/dependence. This setting is experientially real for students in that it utilizes both their mathematical knowledge and their experience with travel as a foundation from which to build more formal mathematics. Further details about the sequence can be found in Wawro,
Rasmussen, Zandieh, Sweeney, and Larson (2012). Furthermore, we emphasize how the RME heuristic of emergent models and Gravemeijer’s (1999) four levels of activity are manifested within the MCR. This is summarized in Table 1; the specific associated details of the MCR as explored within the body of the text.

In Table 1, the left column provides a description of each level of activity, and the right column describes each level’s manifestation within the MCR. Situational activity, which involves students working toward mathematical goals in an experientially real setting, occurs within Tasks 1, 2, and 3. Referential activity involves models-of that refer to physical and mental activity in the original task setting. Tasks are asked without specific reference to the original task setting, but students refer to that setting in solving the tasks. General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. Finally, formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and that no longer requires support of models-for activity.

The first task in the MCR sequence is given to students on the first day of class, prior to any formal instruction. The four tasks that comprise the MCR typically take five to six class sessions to complete. Small group work is alternated with whole class discussions in which students explain their tentative progress, listen to and make sense of other groups’ progress, and come to justified conclusions on the problems. Most tasks within the MCR allow for multiple solution strategies and representations. These aspects are fundamental to the efficacy of the instructional sequence in supporting students’ reinvention of the mathematical concepts. We detail the MCR instructional sequence by summarizing each of the four tasks and providing examples of student work on those tasks. We also describe how each task relates to the four levels of activity within the emergent models heuristic.

Table 1: The MCR sequence summarized via Gravemeijer’s four levels of activity

<table>
<thead>
<tr>
<th>Level of Activity</th>
<th>Manifestation in the Magic Carpet Ride sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Situational Activity</strong></td>
<td>Students explore different ways of combining travel vectors algebraically and geometrically in the Magic Carpet Ride scenario. This occurs within Tasks 1, 2 and 3.</td>
</tr>
<tr>
<td><strong>Referential Activity</strong></td>
<td>Students explore the definitions of linear dependence, linear independence, and span for sets of vectors. Tasks are asked without specific reference to the original task setting, but students refer to that setting in solving the tasks. This occurs within Task 4 and in follow-up questions in Tasks 2, 3, and 4. Students’ organizing activity with the definition and associated concept images of span and linear in/ function as models-of students’ activity in the Magic Carpet Ride setting.</td>
</tr>
</tbody>
</table>
Table 1: The MCR sequence summarized via Gravemeijer’s four levels of activity (continued)

<table>
<thead>
<tr>
<th>Level of activity</th>
<th>Manifestation in the Magic Carpet Ride sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General activity</strong> involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting.</td>
<td>Students make and support conjectures about properties of sets of vectors regarding linear dependence, linear independence, and span. The new mathematical reality of $\mathbb{R}^n$ emerges. Students’ organizing activity with the definitions and concept images of span and linear in/dependence function as models-for enlarging the new mathematical reality of $\mathbb{R}^n$ and making new generalizations about sets of vectors in $\mathbb{R}^n$ that do not refer to the MCR setting.</td>
</tr>
<tr>
<td><strong>Formal activity</strong> involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity.</td>
<td>Students use definitions of span, linear dependence and linear independence without having to unpack the meanings of these definitions (e.g., use definitions to reason about the Invertible Matrix Theorem). Does not occur during Tasks 1-4 of the MCR sequence, but rather occurs during the remainder of the semester on tasks unrelated to the MCR sequence.</td>
</tr>
</tbody>
</table>

**Task 1: Investigating vectors and their properties**
A main goal of Task 1 (see Figure 3) is to promote students’ understanding of vectors and their properties. The instructor uses student work as a starting point for introducing formal notation and language for scalar multiplication, linear combinations, vector equations, and system of equations. In Task 1, students are presented with a scenario in which they are preparing to embark on a journey, and they are given two modes of transportation - a magic carpet and a hover board. They are asked to investigate whether it is possible to reach a certain location - the location where Old Man Gauss lives - with the two modes of transportation. The movement of the magic carpet, when ridden forward for a single hour, is denoted by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to indicate motion along a “diagonal” path resulting from displacement of 1 mile East and 2 miles North. The other mode of transportation, a hover board, is defined similarly along the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The problem as given to students is shown in Figure 2.
Figure 3: Task 1 of the Magic Carpet Ride sequence

With respect to the first of the four levels of activity, Situational activity involves students working toward mathematical goals in an experientially real setting. Within Task 1, this is evidenced as students explore different ways of combining travel vectors in $\mathbb{R}^2$ (the set of all vectors $[x, y]$ with $x$ and $y$ as real numbers) towards the goal of determining how to reach Old Man Gauss. This activity helped students explore the notion of a linear combination of one or two vectors in $\mathbb{R}^2$, including its symbolic and graphical representations. Thus, even at this level students develop symbolic and graphical inscriptions that are models of their thinking.

As students share their work and solution strategies, an opportunity to coordinate geometric and algebraic views of the problem and its solution is given. The class begins by exploring the problem setting; for instance, riding the magic carpet forward for three hours would transport you to 3 miles East and 6 miles North of the starting point, and that could be denoted as $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $3 \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Also, a negative scalar with a vector, such as $-4 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, would be interpreted as traveling "backward" or "in reverse direction" on the hover board for 4 hours, and a fractional scalar would translate to travelling for portions of an hour. By sharing their approaches, students have the
opportunity to see the problem in multiple ways and make new connections between their thinking and other possible approaches. This also allowed the instructor to highlight important ideas and connections. For example, the instructor used student work to make explicit how to transition between vector equations, systems of equations, and various geometric representations. Student work is described in more detail in Wawro et al. (2012).

**Task 2: Reinventing the notion of span**

The goal of Task 2 is to help students develop the notion of span in a two-dimensional setting before formalizing the concept with a definition. Task 2 prompts students to determine whether there is any location where Old Man Gauss can hide so that they would be unable to reach him using the same two modes of transportation from Task 1 (see Figure 4). With respect to the four levels of activity, Task 2 first engages students in situational activity as they work to determine if Old Man Gauss can hide. This activity involves students working to find a way to think about all possible linear combinations of vectors, thus leading to the reinvention of the notion of span. Follow-up questions that the instructor poses regarding span position students to engage in referential activity. Referential activity involves making use of tools, inscriptions, and ideas that refer (implicitly or explicitly) to physical and mental activity in the original task setting. As detailed in this section, this occurs as students referred back to the MCR scenario to reason about the span of non-situational vectors.

---

**THE MAGIC CARPET RIDE: PROBLEM TWO**

**SCENARIO TWO: HIDE-AND-SEEK**

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can’t visit him.

**Are there some locations that he can hide and you cannot reach him with these two modes of transportation?** Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Use your group’s whiteboard as a space to write out your work as you work together on this problem.

*Figure 4: Task 2 of the Magic Carpet Ride sequence*

In spring 2010, after working on Task 2 in small groups, determining if Old Man Gauss could hide became reinterpreted as determining if you could “get everywhere” with the two modes of transformation, and there was disagreement as to whether it was in fact possible to “get anywhere.” Students wrestled, in productive ways, with their interpretations of vector addition, scaling, and linear combinations of vectors and the variety of ways they might appear in geometric depictions. Figure 5 provides two examples of student work on Task 2. In Figure 5(a), Group 1 concluded that, with the two modes of transportation, only locations within the shaded regions, which we refer to as a “double cone,” could be reached; thus, Old Man Gauss could hide anywhere outside of the shaded regions. Group 4 (Figure 5(b)), on the other hand, argued that you could “get anywhere” in the entire plane with the two modes of transportation. Student work is described in more detail in Wawro et al. (2012).
After student approaches had been discussed in class, the instructor introduced the language and formal definition of span in relation to the work and ideas set forth by students. She stated the span of a set of vectors is all possible linear combinations of those vectors, or in other words, all places you could reach with those vectors. Furthermore, any vector that can be written as $c_1v_1 + c_2v_2 + \ldots + c_pv_p$ for some real numbers $c_1, c_2, \ldots, c_p$ is in the span of $\{v_1, v_2, \ldots, v_p\}$.

The instructor’s introduction of a new term and its definition functioned to support students in shifting from situational activity (working within the MCR context) to referential activity (working with the idea of span, which was motivated by students’ prior work of determining if Old Man Gauss could hide).

Next, the instructor asked if Gauss’s location from Task 1, the vector $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$, was in the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, the two modes of transportation. The class decided that it is in the span because there exists scalars, namely 30 and 17, such that $30 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 17 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 107 \\ 64 \end{bmatrix}$. The instructor then posed new questions about span for the students to consider within their small groups. A variety of responses were shared in whole class discussion. For example, the class determined the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not all of $\mathbb{R}^2$ because you cannot “get everywhere” with those two vectors; rather, you can only “travel” along the line through those vectors. The way many students engaged in these was referential because they made reference to the MCR scenario in solving the problems (which were not in the task setting). In addition to responding to the posed questions, a few groups also developed conjectures about span in a more general way. For instance, two groups developed a generalization that two vectors in $\mathbb{R}^2$ that are not scalar multiples span $\mathbb{R}^2$. This indicates an initial movement towards general activity because these groups were reasoning about properties of sets of vectors.

**Task 3: Reinventing linear independence/dependence**

The purpose of Task 3 is to provide an opportunity for students to develop geometric imagery for linear dependence and linear independence that can be leveraged in the development of the formal definitions of these concepts. In Task 3, students are asked to determine if, using three vectors that represent modes of transportation in a three-dimensional world, they can take a journey that starts and ends at home (see Figure 6). The restrictions placed on the movement of these modes of transportation are that the vectors can only be used once for a fixed amount of time represented by the scalars $c_1, c_2,$ and $c_3$. With respect to the four levels of activity, Task 3 first engages students in situational activity as they investigate existence and uniqueness of solutions to homogeneous equations by considering “journeys that begin and end at home.” In
whole class conversations following students’ work on Task 3, the instructor supports students’ engagement in referential activity by posing follow-up questions that allow them to use their experience with the MCR scenario to justify claims about linear in/dependence for non-situational examples.

Figure 6: Task 3 in the Magic Carpet Ride sequence

In Spring 2010, initial progress on Task 3 was made when the class established that a trip that begins and ends at home can be represented by $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. By formulating the problem in this way, students were able to connect their algebraic activity of previous tasks to their work on this problem. Examples of student work are given in Wawro et al. (2012).

Whole class discussion of the various approaches offered insight into how students conceptualized linear combination of the three vectors. The language of “getting back home” came to represent movement along the vectors and how to combine the vectors so that the journey returned to the origin. The instructor labeled the ability to “get back home” with the term linearly dependent and subsequently introduced the formal definition as follows: Given a set of vectors $\{v_1, v_2, ..., v_p\}$ in $\mathbb{R}^n$, if there exists a solution to the equation $c_1v_1 + c_2v_2 + ... + c_pv_p = 0$ where not all $c_1, c_2, ..., c_p$ are zero, then $\{v_1, v_2, ..., v_p\}$ is a linearly dependent set.

Next, the instructor asked if a set that contained the travel vectors from Task 1, namely $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\}$, was linearly dependent. In this discussion, many students did not appeal to the formal definition of linear dependence. Instead, they referenced “getting back home” as how they knew that the vectors were not linearly dependent. To justify why $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\}$ was not linearly dependent, another student demonstrated that by placing these two vectors into the appropriate system of equations, the scalars $c_1$ and $c_2$ would be forced to be zero. The instructor then tagged the set of these two vectors as linearly independent and introduced the formal definition as follows: A set of vectors $\{v_1, v_2, ..., v_p\}$ in $\mathbb{R}^n$ is a linearly independent set if the only solution to the equation $c_1v_1 + c_2v_2 + ... + c_pv_p = 0$ is if all $c_1, c_2, ..., c_p$ are zero. The instructor’s work in tagging with a new definition functioned as a bridge between students’ prior work and the conventional mathematical term. It also supported students’ engagement in referential activity as they engaged in follow-up problems that utilized the new terminology.
Task 4: Creating Examples of Linearly Independent and Dependent Sets

One main goal of Task 4 is to shift students away from situational and referential activity with respect to the Magic Carpet Ride scenario towards a general level of activity with respect to linear in/dependence. In Task 4, students are asked to generate examples of linearly dependent and linearly independent sets of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Task 4 also prompts students to make generalizations, however tentative, regarding conditions for a set of vectors to be linearly independent or dependent (see Figure 10).

<table>
<thead>
<tr>
<th>Linearly dependent set</th>
<th>Linearly independent set</th>
</tr>
</thead>
<tbody>
<tr>
<td>A set of 2 vectors in $\mathbb{R}^2$</td>
<td></td>
</tr>
<tr>
<td>A set of 3 vectors in $\mathbb{R}^2$</td>
<td></td>
</tr>
<tr>
<td>A set of 2 vectors in $\mathbb{R}^3$</td>
<td></td>
</tr>
<tr>
<td>A set of 3 vectors in $\mathbb{R}^3$</td>
<td></td>
</tr>
<tr>
<td>A set of 4 vectors in $\mathbb{R}^3$</td>
<td></td>
</tr>
</tbody>
</table>

Write at least 2 generalizations that can be made from this table

Figure 7: Task 4 in the Magic Carpet Ride sequence

As students completed the chart in Figure 7, they developed various generalizations about linear in/dependence. The four conjectures listed below, which occurred during the spring 2010 semester, represent typical responses.

1. If you have a set of vectors in $\mathbb{R}^n$ where two of the vectors are multiples of each other, then the set is linearly dependent.
2. If any vector in the set can be written as a linear combination of the other vectors, then the set is linearly dependent.
3. If the zero vector is included in a set of vectors, then the set is linearly dependent.
4. If a set of vectors in $\mathbb{R}^n$ contains more than $n$ vectors, then the set is linearly dependent.

The first two were quickly justified by students. The third and fourth conjectures, however, required more debate and justification. For instance, one group suggested that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ was linearly independent, whereas another suggested that there was no possible set of three vectors in $\mathbb{R}^2$ that could be linearly independent. After the class established that not both could be correct, the instructor asked for volunteers to support either conclusion. Some students referred to the formal definition of linear dependence to justify why any set containing the zero vector must be linearly dependent. With respect to the four levels of activity, the class' use of the definition of linear dependence to determine that any set with the zero vector would be linearly dependent is an example of a general level of activity. This is general activity because the students worked with vectors without referring back explicitly to the MCR scenario as they explored properties of sets of vectors in $\mathbb{R}^2$. This was then followed by referential activity in that the class explicitly referred back to the MCR scenario in order to develop a rich understanding of properties of sets that contain the zero vector.
Next, the class discussed whether a group was correct in conjecturing that there was no set of three vectors in \( \mathbb{R}^2 \) that was linearly independent. Justin, a member of that group, supplied an argument to the class to justify this assertion. Later that day, Justin claimed any set of vectors in \( \mathbb{R}^n \) that contained more than \( n \) vectors was linearly dependent. He explained how an analogous argument applies for \( \mathbb{R}^3 \), using \( \mathbb{R}^3 \) as an example of how the argument generalizes to other dimensions. Justin engaged in referential and general activity to develop and justify generalizations. To explain his conjecture that any set in \( \mathbb{R}^2 \) with more than two vectors was linearly dependent, he first referred to the MCR scenario (specifically to activity within Task 2). He then engaged in general activity as he extended his conjecture to sets of vectors in \( \mathbb{R}^n \) that contained more than \( n \) vectors.

**Beyond the Magic Carpet Ride sequence: Formal level of activity**

Formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity. Students use definitions of span and linear in/dependence without having to unpack the meanings of these definitions. We provide one example of formal level of activity. One student, Abraham, during an individual interview at the end of the semester, was asked to explain how he understands the equivalence of the two statements “The number zero is not an eigenvalue of \( A \)” and “The null space of \( A \) contains only the zero vector.” A portion of his response is given:

An eigenvalue definition something’s like, there’s a nonzero \( x \) such that \( Ax = \lambda x \). And so if the number zero…this says ‘not.’ But I always like to think of if it is. So then if it is, if, if \( \lambda = 0 \), then \( Ax = 0 \). And then by the definition of an eigenvector, we can find a nonzero solution…So then if \( \lambda = 0 \), this is saying by the definition, we can find a nonzero solution, such that \( Ax = 0 \). But this is, it’s linear dependent, by definition, because it’s a nontrivial or the only solution is not the trivial solution. There’s a nonzero solution, so it’s linear dependent if it’s [the eigenvalue] zero. And how does this relate to null space for me? … I think of this because if there’s a nonzero solution here, then the null space doesn’t contain only the zero vector.

The underlined portion highlights his use of the definition of linear dependence to reason about how having zero as an eigenvalue of \( A \) implies that the null space of \( A \) contains more than just the zero vector. To do this, he did not need to unpack the meaning of linear dependence, as was done earlier in the semester. He did not need to refer back to the MCR sequence to reason about linear dependence. Rather, he was able to use its definition without unpacking it to support claims about new concepts in linear algebra.

**7. Conclusion**

Within this chapter we detailed an example of design research within undergraduate mathematics. In particular, the results we presented here were a research-based instructional sequence known as the Magic Carpet Ride sequence, which fosters students’ reinvention of span and linear in/dependence, and its underlying theoretical rationale. Our methodological approach to design research is informed by Cobb (2000), which details methodology for classroom teaching experiments. Theoretically, our work is grounded in Realistic Mathematics Education. Within this chapter, we emphasized how the RME heuristic of emergent models and Gravemeijer’s (1999) four levels of activity informed the creation of the MCR. We conclude with a summary of the levels of activity as manifested in the MCR. Within Table 1, the left column provides a description of each level of activity, and the right column describes each level’s manifestation within the MCR. Situational activity, which involves students working toward mathematical goals in an experientially real setting, occurs within
Tasks 1, 2 and 3. In Task 1, students determine how to reach Old Man Gauss. In Task 2, students determine if Old Man Gauss can hide, and in Task 3, students determine how to begin and end a journey at home. In addition, students develop symbolic and graphical inscriptions that are models-of their thinking and that the instructor is able to label with the terminology of the mathematical community such as linear combination, span, and linear in/dependence.

Referential activity involves models-of that refer to physical and mental activity in the original task setting. In the follow-up questions within Tasks 2 and 3, students explore the definitions of span and linear in/dependence. Tasks are asked without specific reference to the original task setting, but students refer to that setting in solving the tasks. Within Task 4, in which students are asked to create examples of sets of vectors, they often refer back to the MCR scenario to create their examples as well as justify why their examples are valid. General activity involves models-for that facilitate a focus on interpretations and solutions independent of the original task setting. In addition to creating examples in Task 4, students are prompted to develop conjectures about linearly in/dependent sets. This is general activity because the students reason about general properties of sets vectors without referring back explicitly to the MCR scenario.

Creating a new mathematical reality of $\mathbb{R}^n$ begins in referential activity as students explore what it means to "get everywhere" or to "get back home." When the instructor tags these activities with formal definitions of span and linear in/dependence, these definitions and the associated concept images become models of student activity in the task setting. This is the beginning of creating the new mathematical reality. When students are asked to work explicitly with the definitions to answer questions about span and linear in/dependence, the students are transitioning to general activity, in which the definitions serve as models for their organizing activity. In general activity students create conjectures using the definitions and new concept images associated with these definitions. This further develops the new mathematical reality of $\mathbb{R}^n$.

Finally, formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and that no longer requires support of models-for activity. Formal activity occurs later in the semester as students are able to use definitions in the service of making other arguments without having to explicitly recreate or reinterpret those definitions. We provided one example of Abraham using the definition of linear dependence to explain a connection between Eigen theory and null space.

Although retrospective analysis of the data is ongoing, initial analysis suggests that using the MCR sequence as an avenue for investigating span and linear in/dependence, which then leads to a need for complex systems of equations, ameliorated our initial issues to a certain extent. Additional future research involves investigating the instructor’s actions, such as how and when to introduce definitions that connect to students’ prior activity, that support shifts in students’ levels of activity.

Key sources

Key sources on the Magic Carpet Ride sequence described in more detail:
Key sources on research that includes analysis of students’ understanding of linear independence and span (main concepts of the Magic Carpet Ride sequence):


Key sources for classroom teaching experiment methodology:


References


Megan Wawro is an Assistant Professor of Mathematics Education in the Department of Mathematics at Virginia Tech in Blacksburg, Virginia. She received a B.A. and M.A in mathematics from Cedarville University and Miami University, respectively, and her Ph.D. in mathematics and science education jointly from San Diego State University and University of California, San Diego. Her research focuses on the learning and teaching of undergraduate mathematics. Her current work explores student thinking and instructional design in linear algebra, as well as methodologies for documenting student reasoning at both individual and collective levels.

Email: mwawro@vt.edu

Chris Rasmussen is Professor of Mathematics Education in the Department of Mathematics and Statistics at San Diego State University. He received an undergraduate degree in mechanical engineering, a master's degree in mathematics, and his Ph.D. in mathematics education at the University of Maryland. His research focuses on the learning and teaching of undergraduate mathematics, with a focus on courses that serve as a transition from students' current ways of reasoning to more formal and abstract ways of reasoning.

Email: Chris.rasmussen@sdsu.edu

Michelle Zandieh is an Associate Professor and Mathematics Program Leader in the Department of Applied Sciences and Mathematics in the College of Technology and Innovation at Arizona State University. She received undergraduate degrees in mathematics and geology, a master's degree in mathematics, and a Ph.D. in mathematics at Oregon State University. Her research focuses on the learning and teaching of undergraduate mathematics, with a focus on student reasoning in courses such as calculus, linear algebra, geometry and transition to proof.

Email: zandieh@asu.edu

Christine Larson is a post-doctoral research fellow at Vanderbilt University’s Peabody School of Education in the Department of Teaching and Learning. She earned her undergraduate and master's degrees in mathematics from the University of Kansas, and her Ph.D. in mathematics education and learning sciences from Indiana University. Her dissertation research focused on instructional design in introductory undergraduate linear algebra, exploring the ways in which student thinking, modeling, and history of mathematics serve to inform the development of inquiry oriented instructional materials in that content domain. Her current work explores teacher learning and institutional supports for effectively scaling up the implementation of inquiry oriented curricula.

Email: christine.j.larson@vanderbilt.edu