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A Method for Using Adjacency Matrices to Analyze the Connections Students Make Within and Between Concepts: The Case of Linear Algebra

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The central goals of most introductory linear algebra courses are to develop students' proficiency with matrix techniques, to promote their understanding of key concepts, and to increase their ability to make connections between concepts. In this article, we present an innovative method using adjacency matrices to analyze students' interpretation of and connections between concepts. Three cases provide examples that illustrate the usefulness of this approach for comparing differences in the structure of the connections, as exhibited in what we refer to as dense, sparse, and hub adjacency matrices. We also make use of mathematical constructs from digraph theory, such as walks and being strongly connected, to indicate possible chains of connections and flexibility in making connections within and between concepts. We posit that this method is useful for characterizing student connections in other content areas and grade levels.

Key words: Adjacency matrices; Connections; Linear algebra; Research method

"Any fact becomes important when it's connected to another"
(Eco, 1989, p. 377).

The central goals of most introductory linear algebra courses are to develop students' proficiency with various matrix techniques, to promote their conceptual understanding of key concepts, and to increase their ability to relate concepts. For mathematicians, the rich relationships between concepts make linear algebra an especially beautiful and elegant subject. For students, however, these rich

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relationships often result in significant difficulties (Dorier, 2000). For researchers, there is a need to further develop methods that are amenable to dealing with a large number of concepts connected in complex ways.

In this article, we present an innovative method using digraph theory and adjacency matrices to organize, unpack, and systematically analyze the concepts evoked by students and the subsequent connections made between those concepts. We distinguish between two types of connections: within-concept connections and between-concept connections. In linear algebra, for example, a student might connect different interpretations of linear independence. This would be a *within-concept connection* because the connection involves different interpretations of the same concept, namely that of linear independence. On the other hand, a *between-concept connection* is made when a student makes a connection that cuts across concepts, for example, making a connection between an interpretation of linear independence and a different concept such as invertibility or span. Our use of adjacency matrices affords insight into both the within-concept and between-concept connections that students make and the qualitatively different ways in which students make connections.

Examining Students' Understanding of Linear Algebra

There is a growing body of research regarding the teaching and learning of linear algebra, the most comprehensive of which is the volume edited by Dorier (2000). This volume highlighted the many difficulties that students have in linear algebra and posited reasons for these difficulties. For example, Hillel (2000) pointed out that one feature specific to linear algebra that tends to cause students difficulties is the existence of three modes of description: the geometric mode, the algebraic mode, and the abstract mode. Students have difficulties within each mode (e.g., describing vectors as both arrows and points in the geometric mode) as well as across modes (e.g., connecting algebraic and abstract modes in order to see distinct strings of numbers as the same vector in different bases). Harel (2000) asserted that student difficulties might also be related to how students are expected to treat vectors and matrices, for example, as objects that can be manipulated and the challenge in adjusting to the associated change in symbolism. For example, in $cx = d$, all three variables take their values from the real numbers, but in the matrix equation $A\mathbf{x} = \mathbf{d}$, A is a matrix and both $\mathbf{x}$ and $\mathbf{d}$ are multicomponent column vectors.

There are also a number of studies that have discussed the difficulties students have with particular concepts in linear algebra. For example, Larson, Zandieh, Rasmussen, and Henderson (2009) examined student understanding of matrices, the equal sign in matrix equations, and informal or intuitive notions of eigenvalues. Studies have been conducted regarding student difficulties with the notions of basis (Hillel, 2000), linear transformation (Dreyfus, Hillel, & Sierpinska, 1999), rank (Dorier, Robert, Robinet, & Rogalski, 2000), linear independence (Bogomolny, 2007; Harel, 1997; Trigueros & Possani, 2013), and span (Stewart & Thomas, 2009). These and related studies that examine student thinking in linear algebra have tended to focus on how students understand the concepts in and of themselves (within-concept connections) as opposed to how students make connections across ideas (between-concept connections).

Although there are few empirical studies focused on between-concept connections, the importance of making between-concept connections can be seen in various policy documents and theoretical reflections. For example, Harel (1997) named the ability to connect ideas as an indicator of understanding a concept. In particular, he pointed out that "one of the most appealing aspects of linear algebra—yet a serious source of difficulty for students—is the 'endless' number of mathematical connections one can (must) create" (Harel, 1997, pp. 111–112). Based on Sierpinska's (2000) work, Dorier and Sierpinska (2001) stated that a tendency toward theoretical rather than practical thinking is one characteristic required for the understanding of linear algebra. Of relevance here is their claim that in theoretical thinking, "reasoning is based on logical and semantic *connections* [emphasis added] between concepts within a system rather than on the 'logic of action,' contingent upon empirical, functional and contextual associations between objects or events" (Dorier & Sierpinska, 2001, p. 263).

Of the few empirical studies that explicitly examine the between-concept connections that students make in linear algebra, concept maps have been the dominant analytic method. For example, Stewart and Thomas (2008) studied the connections students made by asking them to create concept maps linking the notions of span, linear combination, basis, linear independence, and subspace. Some students were able to create concept maps that connected basis with span or linear independence, and some were missing one or both of these links. By analyzing these concept maps in comparison to an a priori genetic decomposition (Dubinsky & McDonald, 2001) of basis, the authors were able to posit potential pedagogical changes, such as focusing on the notion of linear combination more or trying to develop a more embodied view of basis.

Meel (2005) explored how effective student-constructed concept maps might be with respect to analyzing student understanding of particular mathematical concepts. Through three phases of data collection and analysis, Meel had students create concept maps of linear algebra concepts, reflect on what they would change in their concept map given the chance, and in some cases even create a second concept map. He concluded that student-generated concept maps could, at most, provide information about students' evoked concept images for the associated mathematical concepts and their connections at the time of the concept map's creation. He did not endorse their reliability as a tool for providing definitive information about a student's understanding because "for some students, wholesale reorganization of concept maps reveal alternative forms of representation that indicate multiple levels of connections that cannot be represented through concept maps" (Meel, 2005, p. 887).

Similar to Meel (2005), who found concept maps to have some limitations, we found that methods associated with analyzing concept maps (Novak & Gowin, 1984; Williams, 1998) or problem-solution maps (Selvaratnam & Canagaratna, 2008) did not sufficiently help us as researchers to organize the data and compare across students when a large number of ideas were evoked. We note that in the previous studies, the concept maps were student generated and hence did not emerge from a deep analysis of student thinking. We made an initial effort to create concept maps but quickly found that these did not enable us to systematically depict and analyze the overall structure of students' connections in a clear, concise form. Furthermore, concept maps did not allow us to easily identify patterns of within-concept and between-concept connections.

Adjacency Matrices as Analytical Tools

We begin this section with a brief review of the definitions for digraphs and adjacency matrices. A *graph* consists of a finite set of points called vertices and a finite set of lines connecting these vertices called edges. A graph in which direction is indicated for every edge is called a *directed graph* or *digraph* (Ore, 1990). In pure mathematics, graphs are not always studied through their graphical representation; instead, one can study a less visual, more systematic representation of a graph called the adjacency matrix. For a given digraph, its *adjacency matrix* is defined as a square matrix with one row and one column for each vertex; an entry of k in row X and column Y indicates k edges from vertex X to vertex Y, and an entry of 0 indicates that there exists no edge connecting X to Y (Chartrand & Lesniak, 2005). Figure 1 gives an example of a digraph and its corresponding adjacency matrix.

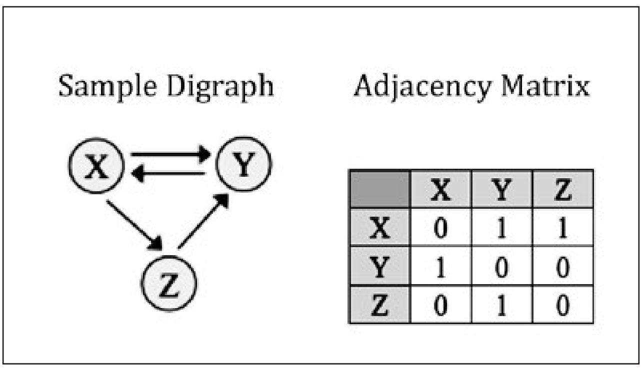


Figure 1. Sample digraph with three vertices and its associated adjacency matrix.

In applied areas of mathematics, digraph theory has a long history as an analytical tool. For example, Harary, Norman, and Cartwright (1965) examined visualization of propositional logic through implication digraphs, and Schvaneveldt, Durso, and Dearholt (1989) used adjacency matrices to study proximity data. A review of the literature reveals that our particular way of using adjacency matrices to analyze student reasoning and justification is relatively innovative in mathematics education. By that, we mean that our use of adjacency matrices differs from prior studies across as least one of the following four dimensions: research goals, data collected, construction of adjacency matrices from the data, or implications of analysis with adjacency matrices. The remainder of this section highlights other uses of adjacency matrices as analytic tools within educational research and how these are similar to and different from the method presented in this article.

In his review of graph theory, Tatsuoka (1986) highlighted some applications of adjacency matrices and digraphs as analytical tools in educational research. The articles Tatsuoka reviewed deal primarily with the use of graph theory in the context of item response theory, focusing on test reliability by developing methods to decrease subjective judgment in constructing item chains, item hierarchies, or clusters. Although we acknowledge the importance of this body of work, we emphasize the categorical distinctions between it and our own work. For instance, one goal of hierarchical structure analysis, according to Tatsuoka, is to determine what knowledge is prerequisite for future learning in order to inform the chronology of curricular enactments. He stated that graph theory is "useful in detecting hierarchical relations among items and thereby aiding in the sequencing of instruction in accordance with procedural skills" (Tatsuoka, 1986, p. 316). In contrast, we used students' verbal explanations of connections between mathematical concepts as data for adjacency matrix analysis. This method facilitated our goal as researchers to provide a nuanced characterization of how students make connections between and within concepts and to identify the overall structure of those connections.

A use of adjacency matrices as analytic tools more similar to our own comes from the field of the learning sciences. Shaffer et al. (2009) developed Epistemic Network Analysis (ENA) as a means to analyze what elements of a profession's epistemic frame (the values, knowledge, skills, and identity associated with that professional community) are developed by those playing games created to simulate practices of that particular profession. ENA makes use of adjacency matrices by treating the various elements of an epistemic frame as the vertices in a graph and by having edges correspond to verbal utterances, coded as epistemic frame elements. For instance, Nash and Shaffer (2011) investigated how learners develop epistemic frames for the profession of urban planning through playing a related game and interacting with professional mentors. Their use of adjacency matrices enabled them to achieve their goal of characterizing the alignment of learners' professional epistemologies with those of mentors within that professional community. Whereas that work characterized learning in a manner more aligned with becoming a member of a community of practice, the method of analysis offered in our research focused explicitly on individual students' mathematical understanding.

A study by Strom, Kemeny, Lehrer, and Forman (2001) used digraphs and adjacency matrices to analyze the development of concepts related to area and area measure within one 52-minute lesson. The authors treated the collective discourse of a fourth-grade class as the unit of analysis, and they segmented the discourse into talk turns, including both teacher and students, and assigned a code to each talk turn. Strom et al. had three code categories: conceptual knowledge, procedural knowledge, and history and notation. They used codes within these categories as vertices, and they used edges to indicate when one code chronologically followed another within the discourse. Their analysis relied primarily on directed graphs to represent the temporal flow of events for the entire lesson and to display the coordination among the categories of concept, procedure, and history throughout the lesson. In their directed graphs, edges did not represent specific utterances but rather the chronological flow of a mathematical argument at the collective level. In what the authors referred to as exploratory work, they touched on using adjacency matrices associated with the various directed graphs to summarize the frequency and centrality of the various vertices within the lesson enactment, as well as to serve as a possible predictive model for individual students' recollection of the classroom mathematical development. In comparison, the directed graphs corresponding to the adjacency matrices that we created do represent specific utterances that explicate the connections being made. Furthermore, although the use of adjacency matrices was exploratory in the Strom et al. work, the analysis presented here makes central the use of adjacency matrices as an analytic tool.

Using Adjacency Matrices to Analyze Connections

This section is divided into three subsections. In the first two sections, we describe the context of our research as it led us to explore the use of adjacency matrices to analyze interview data we collected about students' understanding of within- and between-concept connections of key concepts in linear algebra such as linear independence, determinants, span, invertibility, null space, and pivots. In most texts, the connections between these and other concepts are formalized in one or more theorems. For example, in the text used by the class in our study, key concepts were pulled together in a set of equivalent concept statements and referred to as the Invertible Matrix Theorem (Lay, 2003). In the third section, we discuss how constructs and techniques from digraph theory and adjacency matrices can be utilized as a means of organizing and interpreting what students say and how they relate ideas. In particular, we illustrate a means of translating student utterances into a matrix format and what analysis of those matrices tells us about student reasoning. Relevant definitions from digraph theory and matrix theory are used to give vocabulary to the structure of student thinking.

The Context of Our Research

Data for this analysis come from a semester-long classroom teaching experiment in a linear algebra course at a large southwestern university in the United States (see Cobb, 2000, for methodological details on conducting classroom teaching experiments). There were 33 students in the course, most of whom were engineering or mathematics majors. As part of this classroom teaching experiment, 9 students volunteered to participate in individual, semistructured interviews (Bernard, 1988) conducted at the end of the semester. Each interview was video recorded and lasted approximately 90 minutes. Data included the interview video recordings, complete transcriptions, and copies of all written work produced in the interview.

The interview consisted of several main questions, only one of which is relevant for this article. The interview question we analyzed asked students, given a 3×3 invertible matrix A, to determine whether five different claims are true or false and to explain their reasoning in each case. The question in its entirety is shown in Figure 2. Note that the statements of the prompt are intended to unpack the Invertible Matrix Theorem, a theorem in the text and used in class that links equivalent statements for $n \times n$ matrices. The question prompt in Figure 2 begins with a specific $n \times n$ case, a 3×3 case, and one statement from the theorem (*A is invertible*) as the given supposition. Each of the five claims is based on a different equivalence from the Invertible Matrix Theorem; all claims in Figure 2 are true except for Claim (ii), which would be true if it stated that the determinant is not equal to zero.

> Suppose you have a 3×3 matrix A, and you know that A is invertible. Decide if each of the following statements is true or false, and explain your answer.
>
> (i) The column vectors of A are linearly independent.
>
> (ii) The determinant of A is equal to zero.
>
> (iii) The column vectors of A span $\mathbf{R}^3$.
>
> (iv) The null space of A contains only the zero vector.
>
> (v) The row-reduced echelon form of A has three pivots.

Figure 2. Interview question, showing the given hypothesis and Claims (i)–(v).

In the interview, students were not given Claims (i)–(v) all at once. Instead, they were given Claim (i) with the other claims covered and asked to decide if Claim (i) is true or false and to explain their reasons. After completing this part, Claim (ii) was revealed and discussed and so on. This part-by-part approach was used in order to allow a clearer focus on the student's within- and between-concept connections for each concept. Students typically spent about 30 minutes on this one question.

Students were encouraged to think aloud and were frequently asked follow-up questions aimed at unpacking their interpretations of each concept individually and how connections build from these interpretations. For example, following Claim (i), if a student did not readily explain his or her thinking, the interviewer asked one or more of the following questions as appropriate:

- What does it mean for a set of vectors to be linearly independent/dependent?
- Do you have a geometric way of thinking about the linear independence/dependence?

- What does it mean for A to be invertible?
- How does that relate to what you previously said about linear independence/ dependence?

The interview protocol contained similar follow-up questions for Claims (ii)–(v). In instances where a student's explanation was unclear, such as the vague use of deictic markers such as *it* or *those*, the interviewer also used follow-up questions to help clarify meaning. In some cases, a student would build on a previous response; for example, when discussing Claim (iii), a student referred to Claim (i). In such cases, the student was asked to revisit the earlier response to clarify how he or she was relating concepts.

Analyzing Interview Data

In the spirit of grounded theory (Strauss & Corbin, 1998), data analysis began with reviewing the videos, transcripts, and student written work to understand each student's reasoning. Descriptions were then written for each student. The transcripts and descriptions of all nine interviews were used to create codes for the different interpretations of each concept that students used to explain their reasoning. Direct statements from the Invertible Matrix Theorem or negations of those statements that arose in students' explanations during the interviews were treated as main codes, whereas interpretations of those statements given by students were treated as subcodes of the appropriate main code. For example, after reviewing the interviews for Claim (iv)—*The null space of A contains only the zero vector*—for all nine students, three primary interpretations of null space emerged from the data. The first connects the null space to the solutions of the matrix equation $A\mathbf{x} = \mathbf{0}$. The second interpretation focuses on the solutions to the vector equation $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$, where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are the column vectors of matrix A and x, y, and z are scalars used in the linear combination of vectors. The third and more geometric interpretation of null space examines how to combine the column vectors of A so that their linear combination begins and ends at the origin. Thus, the concept statement *the null space of A contains only the zero vector* from the Invertible Matrix Theorem became main code I, and the three interpretations became subcodes I1, I2, and I3, respectively. Figure 3 shows all codes for concept statements and subcodes for interpretations relating to the null space.

To remain consistent with the students' reasoning, we developed codes for both the positive statement and its negation; code J is the negation of code I, code J1 is the negation of code I1, and so on (see Figure 3). A complete list of all codes for student reasoning on Claims (i) through (v) is given in Appendix A. Similar to code I and its subcodes, all capital letters in the list of codes refer to a concept statement (or its negation) from the Invertible Matrix Theorem, whereas a capital letter followed by a number refers to a specific interpretation of that concept statement (or its negation).

For some concept statements, such as invertibility (code A), span (code G), and

I. Nul $A = \{\mathbf{0}\}$: Null space of A contains only the zero vector

1. *Algebraic – Matrix Equation* – The zero vector is the only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$
2. *Algebraic – Vector Equation* – The zero vector is the only solution to the vector equation $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$
3. *Geometric* – The only way to return to the origin with a combination of the column vectors is when each vector is multiplied by 0

J. Nul $A \neq \{\mathbf{0}\}$; Null space of A contains more than the zero vector

1. *Algebraic – Matrix Equation* – The zero vector is not the only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$
2. *Algebraic – Vector Equation* – The zero vector is not the only solution to the vector equation $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$
3. *Geometric* – Can return to the origin with some nonzero combination of the column vectors

Figure 3. Codes for student reasoning associated with Claim (iv) regarding null space.

pivots (code K), some student interpretations were more procedural than others. For example, some students interpreted the invertibility of matrix A to mean one can use a calculator or row-reduction of the augmented matrix to find the inverse of A. For a given concept statement “X,” incorrect or incomplete interpretations or more procedural interpretations were listed higher in the interpretations underneath each concept statement, usually coded as “X1” or “X2.” Less procedural interpretations of a concept statement were listed lower, typically coded as “X3,” “X4,” or “X5.” As is the case with null space, however, not all interpretations followed this format.

All codes in Appendix A come from concept statements and interpretations given by the students in the course of the interviews. Although a number of the interpretations in the inventory may coincide with what other researchers have found, all coded interpretations originated in statements made by students in this study. Some interpretations in the subcodes are unique to the instructional sequence and thus the students interviewed. For instance, code G2, an interpretation of “the column vectors of A span $\mathbf{R}^3$,” refers to being able to “get to every point in the dimension.” We conjecture that this interpretation of span is strongly tied to a task sequence utilized within this class that supported students’ reinvention of span and linear independence or dependence by building from their intutive notions of travel (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012). The construction of this inventory of brief descriptions of students’ conceptions given in Appendix A is a valuable contribution to what is known about how students conceptualize key ideas in linear algebra.

All connections that students made were coded according to the implication made by the student. For example, DF3 means that the student explained how concept statement D implies interpretation F3. In no case did we infer that a student made a connection when there was not sufficient evidence to do so. The very nature of our coding scheme serves the function of illuminating the underlying structure of the connections that students made. A capital letter by itself (a main code) is a concept statement (or negation thereof) from the Invertible Matrix Theorem, and a capital letter together with a number (a subcode) is an interpretation of the concept statement. As such, combinations such as DF3 or F3J2 provide information about the role of the concept statements and interpretations in forming within- or between-concept connections.

Evidence for the connections that a student made took on three different linguistic forms. The first and most prevalent way that a student made a connection was with an explicit logical implication. Logical implications used either *if-then* phrasing or linking words such as *when*, *because*, *should*, and *so*. Examples of such connections include:

- "And <u>if</u> the determinant is 0, <u>then</u> they're [the vectors are] dependent." (FD)
- "They lie on the same thing [collinear/coplanar vectors], <u>so</u> it [the determinant] would be 0." (D3F)
- "<u>Because</u> in order [for the column vectors] to be [linearly] independent, it [the matrix *A*] has a pivot in each space [points to main diagonal of some unwritten matrix]." (K4C)

A second linguistic form that students used to make connections involved the use of a linking word or phrase for sameness. Such linking words include *means*, *is*, *is the same as*, and *is like*. It is possible that when a student used such an expression, she or he considered the implication to be bidirectional. We did not, however, code such statements as two implications because doing so might be unwarranted. Instead, we erred on the side of caution and coded this as one implication, in the order in which the concept statements or interpretations were uttered. Examples of such connections include:

- "[Linearly] independent <u>means</u> that there aren't any free variables." (CK5)
- "The null space <u>is</u> when you set the matrix equal to the zero vector and see if there's a vector that will get you the zero vector, other than another zero vector." (J1J)

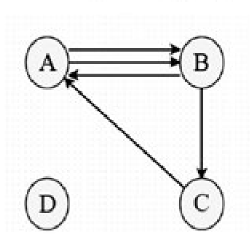
The third way in which students made a connection involved an actual or hypothetical computation. Of the 128 connections that we identified in our analyses, only five were of this form. For example, in order to know if three vectors were linearly independent, one student responded, "The first thing I think of is solving the matrix equation [proceeds to solve the matrix equation to determine the solution]" (CC1). In this example, it is the actual computation that allowed the student to make the connection between linear independence and a unique solution to the matrix equation.

Codes for student reasoning were initially developed by the first author, who then presented her analyses to the research team in weekly meetings. The research team consisted of the two primary investigators of the larger research project and two graduate students. At these meetings, student transcripts, descriptions, and codes were discussed and vetted. In addition, each research team member independently coded particularly difficult sections of transcript, and the research team agreed on a final set of codes. Following the method of Jordan and Henderson (1995), this collaborative, iterative coding process provided multiple occasions to share and defend interpretations of the video and corresponding transcripts, thereby minimizing individual bias by each researcher and eliminating interpretations not grounded in the video. The codes from the transcripts were then used to construct adjacency matrices for individual students.

Constructing Digraphs and Adjacency Matrices

In order to explain how adjacency matrices were constructed out of coded interviews, we revisit the definitions of digraphs and adjacency matrices. Recall that a digraph consists of a set of points called vertices and a set of directed lines connecting these vertices called edges. In our use of digraphs, concept statements and student interpretations of concept statements are the vertices, and edges represent connections made within and between concepts. Also recall that for a given digraph, its adjacency matrix is defined as a square matrix with one row and one column for each vertex; an entry of k in row X and column Y indicates k edges (or connections) from vertex (or concept) X to vertex (or concept) Y, and an entry of 0 indicates that there exists no edge (or explanation) connecting X to Y (Chartrand & Lesniak, 2005). Note that in many contexts, multiple edges from vertex X to vertex Y are prohibited. Because we used digraphs to analyze the connections that students made, possibly on multiple occasions, we made no such restriction, and hence the entries of an adjacency matrix can be any nonnegative integer.

Sample Digraph



Adjacency Matrix M

	A	B	C	D
A	0	2	0	0
B	1	0	1	0
C	1	0	0	0
D	0	0	0	0

Figure 4. Sample digraph with four vertices and associated adjacency matrix *M*.

To illustrate how we used adjacency matrices, consider Figure 4, which shows a digraph and the associated adjacency matrix representing a student's understanding of the relationships between A, B, C, and D. Specifically, the digraph shown in Figure 4 indicates that the student stated a relationship from A to B on two separate occasions. She also made one connection from B to A, one connection from B to C, and one connection from C to A. In the corresponding adjacency matrix, the entry 2 in row A and column B indicates there are two edges, or connections, beginning with A going to B. A similar interpretation can be given for the other entries in the adjacency matrix. Observe that in this sample matrix, A is connected in two different ways to B, and B is connected by one edge to C. Furthermore, there is no directed edge beginning with A leading to C. This is not to say that there is no way to get from A to C; instead, Figure 4 shows that to get from A to C one must go through B first. This idea is also defined in graph theory. A *walk* is a sequence of connected vertices, and the *length of a walk* is the number of edges in that walk.

Adjacency matrices can also be used to clearly identify walks of lengths greater than one by examining powers of the adjacency matrix. If A is the adjacency matrix of a digraph with vertices X and Y, then the number of walks of length n from X to Y is denoted by the entry in row X and column Y of the matrix A^n (Chartrand & Lesniak, 2005). To illustrate this idea of powers of adjacency matrices and the relationship to walks, Figure 5 shows the square of the matrix M from Figure 4. Notice that in row A and column C of matrix M^2 there is an entry 2, indicating that there are two walks of length 2 through which A is connected to C. Thus, an entry n greater than one in a matrix raised to a power greater than one means that there are n different ways of connecting the two concepts.

	A	B	C	D
A	2	0	2	0
B	1	2	0	0
C	0	2	0	0
D	0	0	0	0

Figure 5. Matrix M_2 for the digraph in Figure 4.

Connectivity is another structural characteristic of a digraph that is frequently studied. A digraph is said to be *strongly connected* if for any pair of distinct vertices X and Y, there is a walk connecting X to Y (Minc, 1988). The example digraph in Figure 4 is not strongly connected because there is no walk connecting D to vertex C, or any other vertex in the graph. However, consider the subdigraph on vertices A, B, and C shown in Figure 6. This subdigraph is strongly connected because all vertices are connected to one another. Figure 7(a) shows the adjacency matrix associated with the subdigraph from Figure 6, and Figure 7(b) shows the adjacency matrix after four iterations of multiplication with itself. All entries of matrix M^5

are greater than zero. The significance of this fact is that it indicates a student could connect any concept or concept interpretation to any other concept or concept interpretation through five connections (i.e., a walk of length five) or less. Hence, this is another way to see that the digraph associated with the adjacency matrix is strongly connected. Further implications of what this could mean for analyzing student reasoning are discussed in the following section.

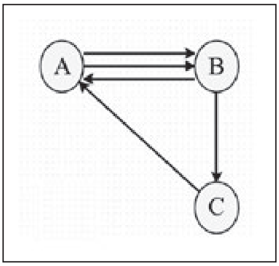


Figure 6. Subdigraph M_1 of the sample digraph from Figure 4.

	A	B	C
A	0	2	0
B	1	0	1
C	1	0	0

Adjacency matrix M_1
(a)

	A	B	C
A	8	8	5
B	6	8	5
C	5	5	4

Matrix M_1^5
(b)

Figure 7. Adjacency matrix M_1 for digraph from Figure 6 and M_1^5 .

Three Types of Adjacency Matrices

After examining the connections made by all 9 students, we selected matrices from 3 students to represent the diversity of ways in which the 9 students made connections. As such, the 3 students selected represent the power of adjacency matrices to capture the different ways in which the 9 students interpreted and connected the concepts in Claims (i) through (v) and structured their connections. The matrix for the first student, Josiah, illuminates his ability to make many different interpretations of concepts and leverage these within-concept connections to make several between-concept connections. As such, Josiah's matrix serves as an example for a *dense adjacency matrix*. The matrix for Bethany illuminates how she mainly made connections between the concept statements themselves and rarely leveraged her interpretations of concept statements to make further connections. As such, she made fewer connections overall, resulting in a *sparse adjacency matrix*. Henry's matrix was chosen to illustrate how some students typically made connections by passing through the same concept and its interpretations to make multiple connections, resulting in a *hub adjacency matrix*. The ways in which the students' connections create these patterns and representative adjacency matrices is explored below in a detailed case report for each of the three students. To show how the codes were used, we have included excerpts of coded transcripts within these reports.

Josiah: Dense Adjacency Matrix

The case of Josiah begins with an interview excerpt illustrating how student statements were interpreted, coded, and represented in an adjacency matrix. Following that discussion, we explain how Josiah's matrix represents a dense adjacency matrix.

In this excerpt Josiah relates the invertibility of the 3×3 matrix A and the span of the column vectors of A.

Josiah: If it's [the matrix A is] invertible, those three vectors [the column vectors of A] should be able to reach any point (AG2). Because you should be able to reach that point with a combination of these vectors and scalars (G2G3). Which is exactly what you're doing with column space (G3G). And then those combinations should be able to create a 3-D object (G3E3). Which is what you're talking about with the determinant (E3E); it's, in my mind at least, three dimensions. And the column vectors being linearly independent is a prerequisite in order to be able to create 3-D objects (CE3). So in a way, their linear independence, it's the same thing as they're spanning $\mathbf{R}^3$ (CG). And the fact that they span $\mathbf{R}^3$ is what allows the determinant to be a nonzero number (GE).

In the previous excerpt, Josiah explains how he understands the connections between invertibility, linear independence, determinants, and span. Specifically, Josiah begins by connecting the invertibility of matrix A with the column vectors of A being able to reach any point in $\mathbf{R}^3$, which we code as AG2 to represent a between-concept connection from concept statement to interpretation G2. He then connects this interpretation to being able to express any point as a linear combination of the column vectors of A. We coded this G2G3 because Josiah connects the geometric idea of being able to reach any point in $\mathbf{R}^3$ with the column vectors of A with being able to express any point algebraically as a linear combination of the column vectors of A. In this example, Josiah used the linking word *because* to make the logical connection between the two ideas. Furthermore, this connection is made between different interpretations of span and thus is considered a within-concept connection. This is signified by both alphanumeric subcodes containing G, the code for the column vectors of A spanning $\mathbf{R}^3$. Furthermore, Josiah connects the previously described algebraic interpretation of linear independence in terms of linear combinations of the column vectors of A, with these same vectors making a nondegenerate three-dimensional object. This three-dimensional object is connected to Josiah's previously described notions of the determinant of A. Thus, he states that the linear independence of the column vectors of A is connected to these vectors spanning all of $\mathbf{R}^3$, which means that the determinant of A is not equal to zero.

The adjacency matrix J_1, listing only those concept statements and interpretations evoked by Josiah in the previous excerpt, is shown in Figure 8. Figure 8 also includes the relevant code descriptions from Appendix A.

	A	C	E	E3	G	G2	G3
A						1	
C				1	1		
E							
E3			1				
G			1				
G2							1
G3				1	1		

A. Matrix A is invertible

C. Column vectors of A are linearly independent

E. det $(A) \neq 0$

  3. Vectors form 3-D parallelepiped with nonzero volume

G. Column vectors of A span $\mathbf{R}^3$ (Column space is $\mathbf{R}^3$)

  2. Can get to any point in $\mathbf{R}^3$
  3. Can make a linear combination for any point in $\mathbf{R}^3$

Figure 8. Adjacency matrix J_1 for Josiah.

For each coded connection in the excerpt, a number is filled in the row corresponding to the first concept in the connection and in the column corresponding to the second concept of the connection. Entries in the adjacency matrix reflect the number of connections between those alphanumeric codes made by the student. In order to make the matrix easier to read, all 0 entries in the matrix are denoted by a blank entry.

Many patterns arise analyzing the connections with the adjacency matrix. We begin by examining entries within the block submatrices on the diagonal of Josiah's matrix J_1. We define *block submatrices* to be the smaller matrices that arise from partitioning the larger matrix wherever the concept statement (the letter) for a group of rows or columns changes. For example, the block submatrices on the diagonal of Josiah's matrix J_1 are outlined in bold in Figure 9. The entries within the block submatrices on the diagonal represent within-concept connections, which are connections between different interpretations of the same concept statement. For example, Josiah shows a flexible understanding of vectors spanning all of $\mathbf{R}^3$ (statement coded G), both geometrically, as reaching all points of the space with these vectors (coded G2), and algebraically, as expressing all vectors in $\mathbf{R}^3$ as a linear combination of the given vectors (coded G3). These connections result in the entries in the diagonal block submatrix in the G rows and columns.

	A	C	E	*E3*	G	*G2*	*G3*
A		1				1	
C				1	2		
E				1			
E3			1				
G			1	1			
G2			1				1
G3				1	1		

Figure 9. Block submatrices on the diagonal of matrix J_1.

There also exist many between-concept connections, denoted by entries within the off-diagonal block submatrices. These connections go between different main concepts signified by different letters in their alphanumeric codes. For example, the 1 in row C and column E3 indicates Josiah's connection between the column vectors of *A* being linearly independent (code C) and these vectors creating a three-dimensional parallelepiped with nonzero volume (code E3). Connections of this type, as illustrated by this example, rely mostly on Josiah's interpretations of the concept statements, in this case, determinants as being the volume of the parallelepiped formed by the column vectors of *A*. Connections between interpretations result in entries not in the upper-left corner of the block submatrices. This pattern indicates that Josiah not only has a firm understanding of how the different concept statements relate to each other but that he also uses his interpretations to relate concept statements and build his between-concept connections.

As an example of the potential of examining powers of the adjacency matrix to see the possible chains of reasoning in a dense adjacency matrix, we consider the adjacency matrix J_1 to the third power, denoted J_1^3. Figure 10 depicts the matrix J_1^3, which indicates the number of walks of length three possible between the listed concepts. This matrix was generated by cubing the matrix J_1.

Among the three walks of length three is the connection between the invertibility of matrix *A* and the column vectors of *A* spanning all of $\mathbf{R}^3$ (AG). This connection is seen through the chain of connections given by Josiah in the first half of the excerpt, repeated below.

> If it's [the matrix *A* is] invertible, those three vectors [the column vectors of *A*] should be able to reach any point (AG2). Because you should be able to reach that point with a combination of these vectors and scalars (G2G3). Which is exactly what you're doing with column space (G3G).

	A	C	E	E3	G	G2	G3
A				1	1		
C							
E							
E3							
G							
G2			1				
G3							

A. Matrix A is invertible
C. Column vectors of A are linearly independent
E. det $(A) \neq 0$
  3. Vectors form 3-D parallelepiped with nonzero volume
G. Column vectors of A span $\mathbf{R}^3$
  2. Can get to any point in $\mathbf{R}^3$
  3. Can make a linear combination for any point in $\mathbf{R}^3$

Figure 10. Josiah's matrix J_1^3.

This connection between invertibility and span, coded AG, may be difficult to identify in the matrix J_1. By taking powers of J_1, we can see how, through a chain of within- and between-concept connections, Josiah could potentially make even more connections than those in the original matrix with walks of length one. For example, the matrix suggests that Josiah could connect A, the invertibility of matrix A, with interpretation E3, that there is a parallelepiped of nonzero volume formed by the column vectors of A. Our intent here is to show one example of the potential for examining the powers of adjacency matrices to elucidate the possible connections students make through longer chains of connections as is typically seen in dense matrices like Josiah's.

Returning our attention back to the matrix J_1, we see that the digraph associated with the matrix is not strongly connected. For instance, consider column A in matrix J_1, which has no entries. This is because Josiah made no connections *to* concept statement A. Thus, the digraph representing Josiah's connections within this excerpt cannot be strongly connected. We note that in the statement of the questions, students were always asked to start by assuming concept statement A, and as a result Josiah may never have felt the need to build connections *to* concept statement A but only to build connections *from* statement A.

We now shift our focus to adjacency matrix J in Appendix B, which denotes all connections Josiah made as he addressed Claims (i)–(v) of the interview question (see Figure 2). For all adjacency matrices, note that the white regions within the matrix represent accurate connections, and entries within the shaded regions indicate incorrect connections. We begin by observing that all the connections Josiah made lie in the unshaded regions of the matrix, which indicate mathematically correct connections.

Josiah's adjacency matrix J serves as our example of a dense adjacency matrix because of its flexible and varied use of interpretations to build several within- and between-concept connections throughout the interview. Several features of matrix J set the matrix apart from the other representative adjacency matrices. For example, the majority of the entries lie within the block submatrices off of the diagonal. These entries indicate that a large portion of Josiah's explanations involve between-concept connections, often built from interpretations of different concepts such as a geometric view of linear independence (C4) with a geometric interpretation of determinants (E3). The between-concept connections frequently use interpretations of the concepts, indicated by the use of subcodes with both a letter (for the concept) and a number (the interpretation of that concept). This pattern can be seen in the adjacency matrix as most entries lying outside of the upper-left corner of each block submatrix. We will later see, in Bethany's interview, examples of between-concept connections that do not leverage interpretations of the concepts, indicated by entries in the upper-left corner of each block submatrix. We note that when Josiah did make connections marked in the upper-left corner of the block matrices, such as the 2 in row C and column G, these between-concept connections relating distinct concept statements were often coupled with between-concept connections leveraging the interpretations seen with entries outside of the upper-left corner of the block submatrices. Thus, Josiah could both connect the concept statements directly and unpack these relationships between concept statements through more detailed interpretations of the concept statements. These patterns identified in matrix J permit us to quickly summarize Josiah's understanding as accurate and full of many interpretations that are readily used to build many connections between all the concepts, thus resulting in a dense adjacency matrix.

Bethany: Sparse Adjacency Matrix

In contrast to the adjacency matrix J, consider Appendix C, which shows the adjacency matrix B resulting from the complete interview on Claims (i) through (v) with another student, Bethany. In contrast to Josiah's matrix, Bethany's adjacency matrix is sparse, has fewer connections, and suggests a more superficial structure of between-concept connections. We begin by observing that large portions of the entries are in the upper-left corners of the block submatrices. These entries in the upper-left corners suggest that several of Bethany's utterances were between-concept connections built by directly connecting concept statements of invertibility, linear independence, determinant, and span or procedural interpretations of these concepts. She often did not use rich, flexible interpretations of these concepts to build connections between them, which would be indicated by entries outside of the upper-left corner seen frequently in Josiah's matrix. As such, her between-concept connections did not leverage interpretations, only relating concept statements.

To illustrate the type of explanation that would result in the entries in the upper-left corners of the block submatrices of Bethany's adjacency matrix B, consider the following excerpt and Figure 11, which corresponds to the excerpt.

Bethany: "The determinant of A is equal to 0." Hm. I think that was true, if it was invertible (AF). . . . And I know, I remember how to solve for the determinant, to see if it's 0. But I think there's some way [pauses for

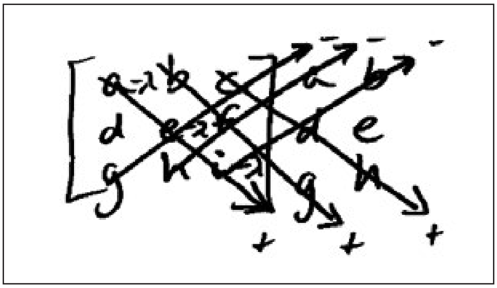


Figure 11. Bethany's procedurally oriented interpretation of determinant.

3 seconds] to augment it, so it all goes to 0, I don't know. Because I know for a 3 by 3. So it would be [writes generic 3 × 3 matrix with variables] that. [Writes first two columns of matrix to the right of the matrix; draws diagonal lines.] I just get confused which. I think these are the ones you added [writes addition signs next to upper-left to lower-right diagonals], and these are the ones you subtracted [writes subtraction signs next to lower-left to upper-right diagonals], but I don't remember. And then you get a value for that (F1F).

Interviewer: OK. What does that value mean to you?

Bethany: That was [pauses 2 seconds]. Oh. I know that if, because I'm trying to think of eigenvalues, but that was if you did subtracting in there [writes "-λ" from diagonal entries of original generic 3 × 3 matrix]. And then solved for those lambdas. But the determinant, I don't really know what it means.

Interviewer: OK. Works for me. But you said that the determinant of A is equal to 0, and if the A is invertible then the determinant is equal to 0. So why exactly did you say that that was the case?

Bethany: From that theorem.

Interviewer: From the Invertible Matrix Theorem?

Bethany: Yup.

In this excerpt, Bethany begins by stating that if the matrix A is invertible, then the determinant of A is equal to zero (code AF). However, when she attempts to unpack her understanding of determinant, she can only interpret the determinant as computational procedure for a 3 × 3 matrix (code F1F). Bethany does not leverage this interpretation to connect invertibility of the matrix A (code A) to the determinant of the matrix (codes E and F). Instead, Bethany makes this within-concept connection and leaves it unrelated to her previous between-concept connections, despite follow-up questions that prompt her to unpack this interpretation further and build connections from this interpretation. Her only reason for making this connection, she cites, is the Invertible Matrix Theorem, the very theorem she is being prompted to unpack.

Furthermore, many of the entries in the upper left-hand corner of the block matrices are found in the row for concept statement A, representing between-concept connections of invertibility to other concept statements. These entries are indicative of true–false responses to the claims, connecting invertibility directly to the concept statement at hand, as with connection AF in Bethany's excerpt. When prompted to unpack this between-concept connection, Bethany did not couple her connection between concept statements with connections leveraging interpretations as Josiah did. Instead, she cited a recollection of the Invertible Matrix Theorem, as seen in the excerpt. This reliance on somewhat vague reference as opposed to a rich relationship between interpretations is a common feature of Bethany's interview and is represented in her matrix by the entries in the upper-left corners without other entries outside of the corner.

Within adjacency matrix *B*, there are some entries that do not lie in the upper-left corners or are found outside of the block submatrices along the diagonal. These entries often indicate connections between linear independence and dependence (concept statements C and D) and the existence or lack of free variables (interpretations K5 and L5). Thus, the matrix *B* also indicates that Bethany did provide some richer connections using the interpretation of a concept statement and not simply by citing the Invertible Matrix Theorem.

We do not include examples of high powers of Bethany's matrix, as her sparse network of connections does not reflect many potential chains of connections. As such, higher powers of Bethany's matrix would not extend our analysis of her understanding.

Finally, it is noteworthy that matrix *B* has two entries in the shaded regions, indicating incorrect connections. Thus, in comparison to concept maps, which do not easily show incorrect connections, the shading used for the adjacency matrices easily does so.

Henry: Hub Adjacency Matrix

Henry represents a third case of how adjacency matrices can be used to illuminate differences in the structure of connections made by students. He made many within- and between-concept connections, much like Josiah and his dense adjacency matrix. In contrast to Josiah, Henry did not make his connections relatively evenly across concepts. Instead, he often returned to the same concept statements and interpretations pertaining to linear dependence and determinants throughout his interview. These concepts became "hubs" for his between-concept connections and can be seen in patterns within his adjacency matrices.

The first instance of using a hub occurred in response to Claim (i), relating invertibility of matrix *A* to linear independence of the column vectors of *A*, and is shown in the subsequent excerpt.

Henry: It says it's invertible, and it's only invertible if they're independent (CA); otherwise, you get a singular matrix (DB). So that would be true. Otherwise you get the divide by zero errors.

Interviewer: What do you mean?

Henry: When you try to take the determinant, if the columns are dependent, because it's like the *ab* plus, no minus *bc* or I can't remember what the order is right now.

Interviewer: Are you thinking of a formula for determinant of the 2 by 2 matrix? That's what it sounds like.

Henry: Yeah. When you divide, you get a zero on the bottom if they're dependent (DB3).

Interviewer: And then you said something about the determinant, so how do you see the determinant playing in?

Henry: Invertible. So if it's invertible, the determinant isn't zero (AE). And if the determinant is zero, then they're dependent (FD). If they're independent, then it's not zero (CE), so that means it's invertible (EA), so (a) is true.

In this excerpt, Henry goes through the hub of determinants to build a between-concept connection relating invertibility with linear independence. Figure 12 shows adjacency matrix H_1, which depicts the connections from the previous excerpt. Of particular interest are the rows and columns for concepts E and F outlined in bold. Note that entries coded E correspond to the determinant of the matrix A is nonzero, while the entries coded F correspond to the determinant of the matrix A is zero. The existence of entries in these rows and columns (and not in the other rows and columns) reflects Henry's tendency to make the connection between invertibility and linear independence not directly but rather through his understanding of the determinant being zero or nonzero. As such, determinants were a hub for the connections he could make between concepts. Determinants were identified as a hub because Henry regularly returned to determinants to make connections throughout the interview. Furthermore, once the connection between invertibility and linear independence was already established, Henry used linear independence as another hub for his between-concept connections.

Appendix D contains adjacency matrix H, which highlights all connections Henry created throughout the course of his entire interview. Note the entries of 2 and 3 and the large number of entries in rows and columns for concepts C, D, E, and F, representing linear independence, linear dependence, nonzero determinant, and zero determinant, respectively. This matrix representing Henry's connections shares many of the patterns seen in J, the matrix representing Josiah's connections. In matrix H, as in the case of Josiah's matrix J, a large portion of the entries lie off of the block diagonal and exclusively in the unshaded regions. However, unlike Josiah's case, the multitude of entries are not spread evenly throughout the matrix but clustered in the rows and columns of specific concepts. The entries in the block submatrices off of the diagonal for concepts C, D, E, and F indicate that Henry used these concepts and interpretations of these concepts to build many within-and between-concept connections. One example of a within-concept connection stemming from

	A	B	B3	C	D	E	F
A						1	
B							
B3							
C	1					1	
D		1	1				
E	1						
F					1		

Figure 12. Adjacency matrix H_1.

interpreting a hub concept is when Henry uses three pens to model three linearly independent vectors, as shown in Figure 13, coded as CC3 in Henry's transcript.

Figure 13. Henry's geometric interpretation of three linearly independent vectors.

This geometric interpretation became the means by which Henry made connections between linear independence and a nonzero determinant. At a later point in the interview, Henry made the same geometric model and explained, "You take the determinant, it'd be taking that out [the three geometrically represented vectors] to make a 3-D parallelogram (C3E3). It's the volume of that (E3E)." Thus, this geometric interpretation led to future between-concept connections with a geometric interpretation of the determinant. Henry's tendency to use a geometric interpretation of linear independence on multiple occasions can also be seen in his matrix (Figure 14) in which a *2* is entered in row C (representing the column vectors of *A* are linearly independent) and column C3 (representing a geometric interpretation of linearly independent vectors).

In contrast to Josiah's case, there are more instances in which, after going through an in-depth chain of connections based on algebraic and geometric interpretations of a concept statement, Henry did not restate these interpretations within the context of a different concept statement. Instead, Henry tended to recall a previous between-concept connection made earlier in the interview, such as the aforementioned connection between linear independence and determinants and simply cited this to make new inferences or meanings. This resulted in several

	A	B	B3	C	C3	C5	D	D3	D4	D5	E	E2	E3	F	G	G2	G3	H	H2	I	I1	I2	I3	J3	K	K2	K3	K5	L4	M
A				1							2				1					1										
B																														
B3																														
C	1				2						1	1	1		1								2		1					1
C3						1							1																	
C5																1														
D		1	1					1	1					1				1	2											
D3							1			1				1																
D4														1																
D5							3																							
E	1			1								1	1																	
E2																														
E3																1														
F		2					3	1	1																					
G											1						1													
G2																														
G3															1															
H																														
H2																														
I																					1	1	1							
I1																														
I2																														
I3																														
J3								1		1																				
K															1											1				
K2																														
K3				2																								1		
K5																														
L4							2			1																				
M																														

Figure 14. Adjacency matrix H_2 for Henry.

entries in the upper-left corner that represent these previously detailed connections later summarized on different occasions.

The multitude of connections Henry made gives rise to the following questions: Did he evoke enough connections to make the digraph corresponding to his adjacency matrix strongly connected? Did he create enough connections to navigate between any two concept statements and interpretations via the walks he generated? Figure 14 shows the matrix H_2, which includes only the rows and columns of concept statements and interpretations for which Henry made at least one connection.

If Henry's connections between concept statements and the interpretations he stated throughout the course of his interview were strong, there would have to be at least one entry in every row and every column of H_2. However, by inspection, we see that there are no entries in the rows for B, B3, G2, H, H2, I1, I2, I3, K2, K5, and M. Thus, not only is H_2 not strongly connected but several more connections would need to be made in order for H_2 to be strongly connected.

As we have noted, Henry's adjacency matrix depicts many chains of reasoning, which would make it a candidate for considering higher powers of the matrix H and thus walks of lengths greater than one. However, we do not gain any greater insight into the hub nature of Henry's connections by considering higher powers of his adjacency matrix because powers of H would simply increase the value of entries in the rows and columns of known hub concepts. For these reasons, we have opted not to include an example of higher powers of Henry's adjacency matrix in this article.

Discussion

We have illustrated the power of adjacency matrices as a methodological tool for detailing the structure of students' within- and between-concept connections. The cases of Josiah, Bethany, and Henry provide examples that illustrate the usefulness of this approach for comparing differences in the structure of the connections as seen in these students' dense, sparse, and hub adjacency matrices, respectively. Furthermore, through adjacency matrices, these connections were analyzed in the context of digraph and matrix theory for mathematical characteristics, such as walks and being strongly connected. These mathematical characteristics indicate possible chains of connections and flexibility in making connections between a larger number of concept statements and interpretations, respectively. Another contribution of the adjacency matrix method is that it requires the construction of a conceptually structured inventory of brief descriptions of students' conceptions (see the list of codes in Appendix A), and this in itself is a valuable result. Such indices of student conceptions could be a useful product for both researchers and instructors.

As evidenced in the data presented, students can develop a large number of connections. In the case of Josiah, the method of adjacency matrices highlighted the abundance of interpretations he exhibited, and these interpretations enabled him to create multiple between-concept connections and justify his reasoning. This structure of multiple, rich connections was signified by a dense adjacency matrix. In the case of Bethany, her sparse adjacency matrix highlighted her relatively few connections, stemming from a reliance on procedural or superficial interpretations of concepts and concept statements. In Henry's case, the clustering of entries in the rows and columns for concept statements C, D, E, and F in his adjacency matrix suggests hubs for his connections, depicting how in the course of Henry's reasoning, he frequently passed through ideas of linear independence, linear dependence, and determinants to make between-concept connections. Thus, for each of our representative students, we found that dense, sparse, or hub adjacency matrices characterized the structure of the connections they made. We leave open the question of whether there are other types of adjacency matrices that could be used to meaningfully model different patterns of connections.

We also highlight the importance of viewing the entire adjacency matrix and all its patterns as a whole. For example, each of the students created connections that are represented with entries in the upper-left corner of the block submatrices, but these entries represented different types of reasoning based on the student. For Josiah and Henry, these entries often reflected restatements of connections previously detailed, whereas for Bethany these represented connections she could not unpack or that she attributed to the Invertible Matrix Theorem without further interpretation. Only in considering the other connections and broader patterns could we distinguish between Josiah's and Henry's restatements of previously unpacked connections and Bethany's recollections of the theorem. It is not merely the density of entries or all entries being within- or between-concept connections that represent strong reasoning but rather a combination of these patterns.

By seeing patterns such as whether or not a student used a rich variety of interpretations or relied on one or more concepts and their interpretations as hubs, we better see the structure of a student's understanding of linear algebra as a whole. This stands apart from previous literature in linear algebra that often focused on within-concept connections rather than a broad network of within- and between-concept connections. As pointed out earlier in the article, policy documents and theoretical reflections on the nature of linear algebra emphasize the complexity (and beauty) of the multiple connections in linear algebra. Beyond linear algebra, many domains in the K–16 curriculum require theoretical thinking that unpacks concepts, builds connections, and uses this network of concepts and interpretations to build an understanding of mathematics as a theory.

Limitations

Although this research shows the usefulness of adjacency matrices to characterize the structure of student connections, there are of course limitations to this method of analysis. For example, when students used a linguistic phrase for sameness, it was unclear whether they intended this to be a bidirectional implication. In order to maintain a low level of inference, only the direction in the order stated by the student was coded. However, there is evidence to suggest some students may have meant the implication in both directions. For example, following Josiah's discussion of Claim (v), the interviewer asked him if he saw any connections between the claims. Josiah responded, "They're obviously all related to the fact that A is invertible. And then assuming that the relationship goes both ways, they should each be related to each other in some way." Josiah's comment suggests some double implication is warranted, but we did not assume bidirectionality implications throughout our analysis, instead choosing to err on the side of caution. With this concern in mind, further refinement of this method coupled with a detailed attention to the language describing the connection in the interview process could lead to improvements in ascertaining a student's intended or unintended bidirectional connections.

Another area for improvement is the extent to which the method can account for students' emerging or partial connections. Moreover, at present the method does not capture qualitative aspects of students' connections, such as a student's inability to explain or any reflective utterances, nor does it capture affective features, such as students' degrees of confidence or certainty. Presently, these are only seen in the interviews themselves and not in the adjacency matrices meant to represent the connections created in the course of the interviews.

Future Directions

As this was an initial foray in the method of adjacency matrices, the goal of this article was to introduce the use of adjacency matrices in analyzing the structure of connections. We see many refinements and extensions of this method that have yet to be examined, and we are eager to see these further explorations taken on within mathematics education research.

One issue for further consideration is the extent to which taking higher powers of the adjacency matrices would indicate connections each student has made or could make, using their interpretations of each concept statement. In the case of using an excerpt from Josiah's interview, higher powers of the matrix indicated between-concepts connections relating concept statements that might otherwise be missed. It is difficult to know, however, which of these entries of the higher powered matrices indicate connections the student actually made in an explanation, such as Josiah did in the excerpt for J_1, versus connections the student could only potentially make. Furthermore, the order in which a student might actually make connections between ideas as she or he works through tasks may be different from that suggested by the higher power of the matrix. These and other ideas regarding higher powers of adjacency matrices require greater in-depth consideration.

Similarly, further consideration should be given to the potential significance of a strongly connected digraph. A strongly connected digraph seems desirable because it would indicate that a student could navigate between any two concept statements or interpretations they evoked. However, a strongly connected digraph of student connections may not be a realistic goal. Although all interviews were conducted to explore students' within- and between-concept connections and many follow-up questions provided opportunities to articulate their understanding, this interviewing process cannot be exhaustive. Furthermore, throughout the interviews students sometimes used the negation of a concept statement, even though the negation was not stated in the task. To stay true to what students actually said, our coding scheme included each claim statement and its negation. This coding decision may have resulted in a matrix that suggests a student is missing some connections, when in reality that may not be the case. This does not impede our analysis, as we analyzed the matrices in comparison to matrices of other students (who were asked the same set of questions) rather than to a matrix with an a priori exhaustive set of connections. It does, however, limit the likelihood of even students with dense adjacency matrices from having strongly connected digraphs. As such, future studies are needed to address these concerns and consider instead measures such as how far a matrix (or submatrix) is from being strongly connected.

Additional directions for ongoing and future research involve using adjacency matrices to analyze the development of mathematical meaning over time, to coordinate individual understanding and collective classroom discourse, and to explore additional quantitative measures that can be paired with adjacency matrices in meaningful ways. Building on the work presented here, Wawro (2011, 2014) has already made progress in these directions. For instance, Wawro (2011) used adjacency matrices and Toulmin's (1969) model of argumentation to analyze the development of mathematical meaning both for the classroom community and for individual students over time. Using the same analytical methods on both units of analysis allows for targeted comparison and coordination of mathematical reasoning, such as if certain within-concept and between-concept connections are salient for a classroom community but not for individual members of that community. Wawro (2014) highlighted aspects of the benefits of adjacency matrix analysis for the classroom community over time through various quantitative measures, such as centrality (Strom, Kemeny, Lehrer, & Forman, 2001), density, and continuity of ideas in linear algebra throughout the semester.

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APPENDIX A

List of Codes

A. *A* is invertible

1. *Calculator inverse* – Can use calculator to produce inverse
2. *Row-reduce to find inverse* – Involves augmenting the matrix with the identity
3. *Formula: Inverse matrix: no "divide by 0 errors"* – Can calculate inverse with a formula, using determinant; relies on the 2×2 inverse matrix formula
4. *Input-output*: "*don't lose information*" – Given output, can use inverse matrix to find input
5. *Can reproduce A* – "Reproduce matrix *A*"; based on class activity

B. *A* is noninvertible/singular

1. *Calculator error* – Can use calculator to see if there is inverse
2. *Cannot row-reduce to find inverse* – Involves augmenting matrix with identity
3. *Formula: Inverse matrix*: "*divide by 0 errors*" – Cannot calculate inverse with formula using determinant; relies on the 2×2 inverse matrix formula
4. *Input-output*: "*lose information*" – Given output, cannot find original input
5. *Cannot reproduce A* – "Cannot reproduce matrix *A*"; based on activity completed in class

C. Column vectors of *A* are linearly independent

1. *Unique solution to matrix equation/system of linear equation*
2. *Geometric – No "losing a degree of freedom"* – Based on class activity
3. *Geometric* – Vectors are *not collinear or coplanar*
4. *Proportional-algebraic* – No vector is a scalar multiple of another vector
5. *Linear combination – algebraic* – No vector is a linear combination other vectors

D. Column vectors of *A* are linearly dependent

1. *No unique solution to matrix equation/system of linear equation*
2. *Geometric* – "*Lose a degree of freedom*" – Based on class activity
3. *Geometric* – Vectors are *collinear/coplanar*
4. *Proportional-algebraic* – One vector is a scalar multiple to another vector
5. *Linear combination – algebraic* – One vector is a linear combination of other vectors

E. det(A) $\neq$ 0: Determinant of A is nonzero

1. *Procedure used to calculate det*(A) $\neq$ 0: Can use Rule of Sarrus/diagonals or Leibniz/cofactor formula to calculate the determinant of a specific matrix A
2. *Geometric – 2-D parallelogram with nonzero volume* – True for 2 × 2 matrix
3. *Geometric – 3-D parallelepiped with nonzero volume* – True for 3 × 3 matrix

F. det(A) = 0: Determinant of A is equal to 0

1. Pr*ocedure used to calculate det*(*A*)*=0*: Can use Rule of Sarrus/diagonals or Leibniz/cofactor formula to calculate the determinant of a specific matrix A
2. *Geometric – 2-D parallelogram with no/zero volume* – True for 2 × 2 matrix
3. *Geometric – 3-D parallelepiped with no/zero volume* – True for 3 × 3 matrix

G. Column vectors of A span $\mathbf{R}^3$

1. *3 × 3 sized matrix* – Size of matrix dictates whether matrix spans; n vectors for n dimensions
2. *Geometric – can get to every point in the dimension*
3. *Algebraic – Linear combination for all points in* $\mathbf{R}^3$

H. Column vectors of A do NOT span $\mathbf{R}^3$

1. *Non 3 × 3 sized matrix* – Size of matrix dictates it cannot span; do not have n vectors for m dimensions
2. *Geometric – Cannot get to every point in the dimension*
3. *Algebraic – No linear combinations for some points in* $\mathbf{R}^3$

I. Nul $A = \{\mathbf{0}\}$: Null space of A contains only the zero vector

1. *Algebraic – Matrix Equation* – The zero vector is the only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$
2. *Algebraic – Vector Equation* – The zero vector is the only solution to the vector equation $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$
3. *Geometric* – The only way to return to the origin with a combination of the column vectors if when each vector is multiplied by 0

J. Nul $A \neq \{0\}$; Null space of A contains more than the zero vector

1. *Algebraic – Matrix Equation* – The zero vector is the not only solution to the matrix equation $A\mathbf{x} = \mathbf{0}$

2. *Algebraic – Vector Equation* – The zero vector is the not only solution to the vector equation $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}$
3. *Geometric* – Can return to the origin with some nonzero combination of the column vectors

K. Row-reduced echelon form (RREF) of *A* has 3 pivots

1. *1s on diagonal* – RREF of the matrix has all 1s on the main diagonal
2. *RREF is the identity matrix* – Can row-reduce to the identity matrix
3. *Pivot in each row* – Includes no row of all 0s
4. *Pivot in each column* – Includes no column of all 0s
5. *No free variables* – Every variable is defined in system of linear equations or matrix equation

L. Row-reduced echelon form of *A* has less than 3 pivots

1. *Not all 1s on diagonal* – RREF of the matrix does not have all 1s on the main diagonal
2. *RREF is the not identity matrix* – Cannot row-reduce to the identity matrix
3. *No pivot in each row* – Includes a row of all 0s
4. *No pivot in each column* – Includes a column of all 0s
5. *Free variables exist* – Every variable is not defined in system of linear equations or matrix equation

M. Other (e.g., Henry's matrix includes an entry in CM, because he explains two linearly independent vectors means "can't get two [vectors] to equal.")

APPENDIX B

Adjacency Matrix J for Josiah

	A	A1	A2	A3	A4	A5	B	B1	B2	B3	B4	B5	C	C1	C2	C3	C4	C5	D	D1	D2	D3	D4	D5	E	E1	E2	E3	F	F1	F2	F3	G	G1	G2	G3	H	H1	H2	H3	I	I1	I2	I3	J	J1	J2	J3	K	K1	K2	K3	K4	K5	L	L1	L2	L3	L4	L5	M
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APPENDIX C

Adjacency Matrix B for Bethany

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APPENDIX D

Adjacency Matrix H for Henry

	A	A1	A2	A3	A4	A5	B	B1	B2	B3	B4	B5	C	C1	C2	C3	C4	C5	D	D1	D2	D3	D4	D5	E	E1	E2	E3	F	F1	F2	F3	G	G1	G2	G3	H	H1	H2	H3	I	I1	I2	I3	J	J1	J2	J3	K	K1	K2	K3	K4	K5	L	L1	L2	L3	L4	L5	M
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