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Advancing Mathematical Activity: A Practice-Oriented View of Advanced Mathematical Thinking

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The purpose of this article is to contribute to the dialogue about the notion of *advanced mathematical thinking* by offering an alternative characterization for this idea, namely advancing mathematical activity. We use the term *advancing* (versus *advanced*) because we emphasize the progression and evolution of students' reasoning in relation to their previous activity. We also use the term *activity*, rather than *thinking*. This shift in language reflects our characterization of progression in mathematical thinking as acts of participation in a variety of different socially or culturally situated mathematical practices. For these practices, we emphasize the changing nature of students' mathematical activity and frame the process of progression in terms of multiple layers of horizontal and vertical mathematizing.

Mathematics education research dealing with the learning and teaching of undergraduate mathematics is an emerging area of interest. Some have considered the

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area of undergraduate mathematics education to be different from K–12 mathematics education because there are more opportunities for what might be thought of as "advanced mathematical thinking." What constitutes advanced mathematical thinking, however, continues to be debated. Does this phrase mean thinking about advanced topics? Does it mean thinking in "advanced" ways about any mathematics? Might it mean something different?

Tall (1992) described advanced mathematical thinking as composed of two components—the specification of concepts by precise mathematical definitions (including statements of axioms) and the logical deductions of theorems based upon them. In addition, Tall stated, "In taking students through the transition to advanced mathematical thinking, we should realize that the formalizing and systematizing of the mathematics is the final stage of mathematical thinking, not the *total activity*" (pp. 508–509, emphasis added). We agree with Tall on this point, and in our research we seek ways to characterize students' total activity as they progress in their mathematical sophistication.

The purpose of this article is to contribute to the dialogue about the notion of advanced mathematical thinking. In particular, we offer an alternative characterization of advanced mathematical thinking that focuses on important mathematical practices and qualitatively different types of activities within these practices. Our characterization of advanced mathematical thinking, that we refer to as *advancing mathematical activity*, is not limited to specific grade or content levels.

We use the term *advancing* rather than *advanced* because we address the process of students' total activity rather than just the "final stage" referred to by Tall. This shift from characterizing "advanced" as a final state to characterizing advanced as a relative term illuminates aspects of students' progression and evolution of reasoning, in relation to their previous activity. Our emphasis on advanc*ing*, rather than on advanc*ed*, also limits the evaluative nature that often comes with the term advanced. In particular, we refrain from characterizing individuals as "advanced" or "not advanced." In our opinion, such characterizations minimize the potentials for all learners, not just the few in upper-level undergraduate courses, to progress in their mathematical sophistication.

We also use the term *activity*, rather than *thinking*. This shift in language reflects our characterization of progression in mathematical thinking as acts of participation in a variety of different socially or culturally situated mathematical practices (Lave & Wenger, 1991; Sfard, 1998; Wenger, 1998). Students' symbolizing, algorithmatizing, and defining activities are three examples of such social or cultural practices. These three mathematical practices are not meant to be exhaustive, but represent a useful set of core practices that cut across all mathematical domains. Another significant mathematical practice, one that we leave to later analysis, is justifying.

The term *thinking* is often used, from a psychological point of view, to describe mathematical growth. Although this focus on thinking often provides useful insights into inferred cognitive structures, it can result in neglecting the types of mathematical activities and ways of participating in these activities that foster and promote progressively sophisticated mathematical reasoning. Because we view learning as acts of participation in different mathematical practices, we intentionally use the term *activity* rather than *thinking*. Our use of the term activity, however, does not reflect a dichotomy between thinking and doing but rather intends to encompass both. We view the relationship between doing and thinking to be reflexive in nature, not dichotomous. As students engage in particular activities, they not only enact their understandings but also enlarge their thinking and ways of reasoning in the process. This is what we mean when we say that the students' symbolizing, algorithmatizing, and defining activities encompass both doing and thinking.

To summarize, our use of the term *activity* reflects a view that mathematics is first and foremost a human activity (Freudenthal, 1991), in which doing and thinking are dualities situated within particular social or cultural practices. As argued by Cobb and Bowers (1999), the notion of participating in practices

> is not restricted to face-to-face interactions with others. Instead, all individual actions are viewed as elements or aspects of an encompassing system of social practices and individuals are viewed as participating in social practices, even when they act in physical isolation from others. (p. 5)

In building on the work of theorists such as Cobb and Bowers (1999) and Lave and Wenger (1991), our efforts in this paper are in line with Tall's (1991) statement, "in trying to formulate helpful ways of looking at advanced mathematical thinking, it is important that we take a broad view and try to see the illumination that various theories can bring" (p. 21). In the sections that follow, we first develop the notion of advancing mathematical activity as acts of participation in different mathematical practices by adapting and modifying Treffers' (1987) constructs of horizontal and vertical mathematizing. We then describe the research projects from which we draw examples. Next, we illustrate and clarify our constructs of horizontal and vertical mathematizing with examples of students' symbolizing, algorithmatizing, and defining activities. In the final section we discuss the links and parallels between the three practices with respect to horizontal and vertical mathematizing, and conclude with some remarks about the utility of these notions for improving mathematics education.

ADVANCING MATHEMATICAL ACTIVITY

Mathematical learning means participating in different types of mathematical practices. To explicate important variations within each practice, we modify Treffers' (1987) idea of progressive mathematizing. Treffers describes progressive mathematizing in terms of a sequence of two types of mathematical activity—horizontal mathematizing and vertical mathematizing. We emphasize that, like doing and thinking, we view horizontal and vertical mathematizing as reflexively related, not as dichotomies. As we make clear in the discussion that follows, the distinction between horizontal and vertical activity is a relative one, one that cannot be made without the other. This reflexivity is a strength because it enables us to make comparisons about the nature of students' activity and it provides us with a language in which to talk about the process by which students develop new views and sensitivities.

According to Treffers (1987), horizontal mathematizing is described as "transforming a problem field into a mathematical problem" (p. 247). This notion of horizontal mathematizing suggests that, for Treffers, what constitutes a problem field is nonmathematical (i.e., some context related to a real-world situation). We treat horizontal mathematizing more broadly to include problem fields or situations that are, from the perspective of those involved, already mathematical in nature. In our view, what constitutes a problem field or problem situation depends on the background, experiences, and goals of those engaged in the mathematical activity. Thus, what constitutes a problem situation for learners in a real analysis course is potentially different from that for learners in an elementary school classroom. Our stance on the relativity of what might be taken as the context for horizontal mathematizing is certainly not a new idea. Dewey (1910/1991) posited that the distinction between what is concrete and what is abstract is relative to the intellectual progress of the person. Indeed, we find the notion of a "final stage" unhelpful in thinking about students' mathematical development because no matter what the intellectual progress, there is always room for growth.

In our broadening of what is meant by a problem field, *horizontal mathematizing* refers to formulating a problem situation in such a way that it is amenable to further mathematical analysis. Thus, horizontal mathematizing might include, but not be limited to, activities such as experimenting, pattern snooping, classifying, conjecturing, and organizing.

Given the situatedness of horizontal mathematizing, vertical mathematizing is then only understood in relation to students' current activity. *Vertical mathematizing* consists of those activities that are grounded in and built on horizontal activities. Thus, vertical mathematizing might include activities such as reasoning about abstract structures, generalizing, and formalizing. Students' new resulting mathematical realities may then be the context for further horizontal mathematizing. To clarify, vertical mathematizing activities serve the purpose of creating new mathematical realities for the students. These new mathematical realities can then be the context or ground for further horizontal and/or vertical mathematizing activities, producing a sequence or chain of progressive mathematizations.

Thus, progressive mathematizing can involve multiple layers of horizontal and vertical mathematizing activities. In the simplest sense, progressive mathematizing refers to a shift or movement from horizontal activities to vertical activities. This shift is not necessarily uni-directional, as vertical activities often "fold back" (Pirie

& Kieren, 1994) to horizontal activities. In more complex cases, progressive mathematizing refers to the fact that students' newly formed mathematical realities, resulting from previous mathematizing, can be the context for additional horizontal and/or vertical mathematizing. This more complex aspect of progressive mathematizing is touched on in the algorithmatizing and defining sections.

Another, and perhaps more significant modification we make to the ideas of horizontal and vertical mathematizing, and one that has been implicit in the preceding discussion and that we now make explicit, is framing progressive mathematizing not in terms of particular ideas such as fractions or long division, as Treffers (1987) does, but in terms of socially and culturally situated mathematical practices. In our view, the mathematical practices of symbolizing, algorithmatizing, and defining are mechanisms by which particular ideas such as fraction, long division, solutions to differential equations, or triangle evolve. This is a nontrivial modification because it calls for attention to the types of activities in which learners engage for the purpose of building new mathematical ideas and methods for solving problems.

In summary, the notion of advancing mathematical activity is the building and progression of practices. Participation in these practices, and changes in these practices, is synonymous with learning (Cobb & Bowers, 1999; Lave & Wenger, 1991). The process by which these practices build and progress is referred to as progressive mathematization, with its multiple layers of horizontal and vertical types of activity. As alluded to earlier, the separation of mathematical activity into horizontal and vertical aspects is somewhat artificial, as in reality the two activities are closely related. However, for the purposes of clarifying the nature of advancing mathematical activity and its progression, this distinction proves useful. As described in the next section, our tightly integrated research, teaching, and instructional design work has provided a unique setting from which the construct of advancing mathematical activity has grown.

RESEARCH SETTING

We have emphasized in our classroom-based research in undergraduate mathematics education the idea of progressive mathematizing and we therefore draw on examples from different classroom teaching experiments (two in differential equations at a mid-sized public university and one in Euclidean and nonEuclidean geometry at a large public university[1]) to illustrate the notion of advancing mathematical activity. The methodological approach we took in these research efforts is that of the class-

[1]The researchers who participated in some or all of the differential equations teaching experiments were Karen King, Chris Rasmussen, Michelle Stephan, and Erna Yackel. The researchers that participated in the geometry teaching experiment were Barbara Edwards, Libby Krussel, Chris Rasmussen, and Michelle Zandieh.

room teaching experiment, as described by Cobb (2000). Data consisted of videorecordings of each class session, videorecorded interviews with individual students, copies of students' written work, and records of project meetings. These classroom teaching experiments had two overarching goals. One goal was to develop paradigmatic case studies of the processes by which students develop particular mathematical ideas in relation to the norms and practices established in their classroom communities. A second related goal was an interest in examining the viability of adapting to the university setting instructional and curriculum design approaches that have been effective for promoting student learning of K–12 mathematics.

In all of these teaching experiments we paid particular attention to the social aspects of the classroom, using Cobb and Yackel's (1996) interpretive framework for coordinating sociological and psychological points of view. Various sociological aspects of this research, including classroom norms pertaining to explanation and classroom mathematical practices, are reported elsewhere (see Rasmussen, Yackel, & King, 2003; Stephan & Rasmussen, 2002; Yackel, Rasmussen, & King, 2000). To a large extent in the differential equations teaching experiment and to a smaller extent in the geometry teaching experiment, the research team's instructional design efforts were grounded in the instructional design theory of Realistic Mathematics Education (Freudenthal, 1973; Gravemeijer, 1994, 1999). The interpretive framework for making sense of the complexity of the classroom learning environment and the instructional design theory of Realistic Mathematics Education were critical to the success of these classroom teaching experiments.

We use examples from our teaching experiments to illustrate and clarify our theoretical development of the notion of advancing mathematical activity, bringing to the fore aspects of horizontal and vertical mathematizing activities within the practices of symbolizing, algorithmatizing, and defining. Rather than primarily viewing mathematics as a set and preorganized discipline that is carefully articulated to students, these three mathematical practices constitute a key collection of activities through which learners can create, organize, and systematize mathematics.

SYMBOLIZING

To better understand how symbolizing can be viewed as a mathematical practice with horizontal and vertical mathematizing aspects, it is useful to first consider some general orienting comments on the nature of symbolizing and symbols. The perspective we take toward symbolizing both departs from and connects with the way that Herscovics (1996) described symbolizing (Rasmussen, 1999). Herscovics wrote that symbolizing

> provides the means to detach a concept from its concrete embodiments. However, the introduction of symbols can be premature if an adequate intuitive basis is lack-

> ing…. Thus mathematical notation can be meaningful only when it is used in the process of *mathematizing* previously acquired informal knowledge. (p. 358, emphasis added)

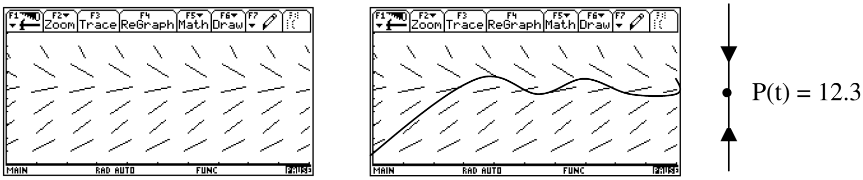
We connect with Herscovics' suggestion that symbolizing is a key aspect of mathematizing and we elaborate horizontal and vertical aspects of such activity. However, instead of viewing symbolizing as a means to "detach a concept from its concrete embodiments" (Herscovics, 1996, p. 358), that is reminiscent of perspectives that treat one's reasoning about mathematical concepts and their representations as separate, distinct entities, we view students' conceptual development and the activity of symbolizing as reflexively related (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Meira, 1995). In this approach, "it is while actually engaging in the activity of symbolizing that symbolizations emerge and develop meaning within the social setting of the classroom" (Gravemeijer et al., 2000, p. 235–236). From this point of view, the need for notation and symbolism arises in part as a means to record reasoning and serves as an impetus to further students' mathematical development. In this way, symbolizing is less a process of detachment and more a process of creation and reinvention. Further mathematizing activity and powerful use of conventional symbols emerge from and are grounded in students' previous symbolizing activities.[2]

In the following paragraphs excerpts taken from a classroom teaching experiment in a university-level differential equations course are used to illustrate aspects of advancing mathematical activity within the practice of symbolizing. In these examples, the symbolizing activities in which students' engage shift from recording and communicating their thinking to using their symbolizations as inputs for further mathematical reasoning and conceptualization. This progression in symbolizing is reflective of the horizontal to vertical mathematizing progression and exemplifies our notion of advancing mathematical activity.

Symbolizing: Horizontal Mathematizing

In the first example students were analyzing solution functions to the differential equation $\frac{dP}{dt} = 0.6P\left(1 - \frac{P}{12.3}\right)$. Their primary tool at this point was the slope field, such as the one shown in Figure 1a. Part of the discussion between the teacher and students focused on the possibility of a particular solution function graph, such as that shown in Figure 1b.

[2]This perspective on symbolizing is compatible with Nemirovsky's (1994) notion of symbol-use, that refers to the actual use of mathematical symbols by someone, for a purpose, and as part of a chain of meaningful events.



(a) (b) (c)

FIGURE 1 Slope field for $dP/dt = 0.6P(1 - P/12.3)$.

Students reasoned that such a solution function was not possible. The following reasoning was typical of the explanations offered by them.

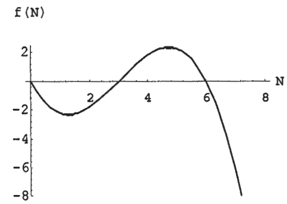
> Joe: I don't think the function would oscillate because if it did then after the function was bigger than 12.3 the slope would still be positive, but from the differential equation and the slope field I know that the slopes are negative when you're above 12.3. So that can't happen.

After further discussion and elaboration, what experts in differential equations understand to be a (partial) phase line was intentionally introduced by the teacher to record students' reasoning, as shown in Figure 1c. From the students' perspective, what the teacher sketched did not yet have a name and was simply a notational device that was consistent with their mathematical reasoning about the behavior of solution functions for the given differential equation. Thus, the symbolizing activity, that was initiated on the part of the teacher in response to student reasoning, is horizontal in nature because this activity seeks to formulate symbolically the given problem situation and students' mathematical reasoning within this situation.

The next example we discuss is a problem given to students on an exam. The task (shown in Figure 2) was a novel one for students, as they had not previously experienced problems of this form. Thus, the work of a student named Kevin[3] that we provide and the symbolizing that this student used are unlikely to be the result of a memorized procedure. This example is different from the previous example in two ways. First, the phase line is now a student record rather than a teacher record, indicating ownership of the inscription. Second, the phase line is used to communicate the long-term behavior of several solution functions, rather that just one solution function. The use of the phase line to communicate the behavior of many solution functions (i.e., the structure of the solution space) is important because it

[3]All names are pseudonyms.

Suppose a population of Nomads is modeled by the differential equation $\frac{dN}{dt} = f(N)$. The graph of *dN/dt* is shown below (CAUTION: this is NOT a graph of a solution function, it is a graph of the right-hand side of the autonomous differential equation)



Graph of *f*(*N*) vs. *N*

For the following values of the initial population, what is the long-term value of the population? Be sure to briefly explain your reasoning.

(i) $N(0) = 2$, (ii) $N(0) = 3$, (iii) $N(0) = 4$, (iv) $N(0) = 7$

FIGURE 2 Student response on novel task.

provides the background for vertical mathematizing in which students use the phase line to infer changes to the structure of the solution space.

The task was to determine the long-term value of a population for various initial populations given a particular differential equation. The novel aspect of the task was that students were provided only with a graph of the differential equation, rather than the equation itself. This circumvented the direct use of a slope field to symbolize and/or reason about the situation and provided an opportunity for them to symbolize (if they so chose) the situation in ways meaningful to them.

As shown in Figure 3, Kevin began his response by interpreting the given information. For example, he wrote, “when the population of nomads is less than 3, the rate of change of nomads with respect to time will decrease [sic, is negative] until they become extinct.” He made similar statements about other ranges for the initial population and then made specific conclusions about the long-term value of the population for specific initial populations.

As shown in Figure 3, Kevin used a phase line in his response. We interpret his use of a phase line as a means to formulate, record, support, and communicate his reasoning. As such, we view this as another example of horizontal mathematizing. We infer that, for this student, the phase line signified the evolution of all solution functions. In this way, the phase line began to take on conceptual meaning independent of the population context. The next example is illustrative of how symbols such as the phase line, that in this case originated as a means to record student reasoning, can shift in function with students’ progressive mathematizing activities.

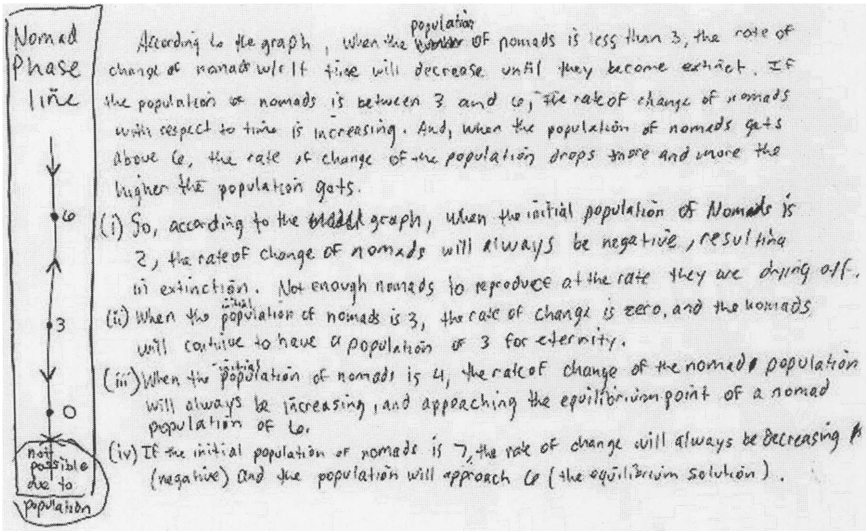


FIGURE 3 Student response.

Symbolizing: Vertical Mathematizing

In the following example we illustrate a vertical mathematizing aspect of symbolizing. In particular, we discuss how this particular student, Joaquin, used results of previous symbolizing activities as input for other symbolizing activities in a dynamic and relational manner. Joaquin's symbolizing activity built on the prior activity with the phase line to generate new mathematical ideas. The vertical nature of this student's work, when contrasted with the previous two symbolizing examples, comes into full view.

The problem, that appears in the textbook by Blanchard, Devaney, and Hall (1998), asks students to identify the bifurcation values of α for the differential equation $\frac{dy}{dt} = y^6 - 2y^4 + \alpha$ and to describe the bifurcations that take place as α increases. At the start of his response, Joaquin noted that whenever the rate of change equation is negative, $y(t)$ will decrease and whenever the rate of change equation is positive, $y(t)$ will increase. This conclusion results in his symbolizing four different regions as shown in Figure 4 where $y(t)$ is either increasing or decreasing.

Our interpretation of his reasoning is that he conceptualized the space of solution functions as being partitioned into four regions separated by three equilibrium solution functions. His references to *y(t)* and his informal use of notation signifying various solution functions $y(t)$, as shown in Figure 4, support this interpretation.

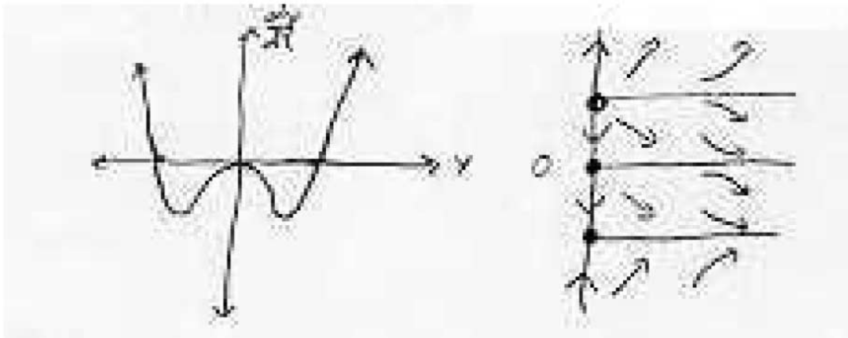


FIGURE 4 Student response on bifurcation task.

At this point, we characterize Joaquin's symbolizing in Figure 4 as horizontal mathematizing because the various representations he used serve to essentially formulate and communicate the problem situation when $\alpha = 0$. Symbolizing is used to create a model-of the situation (Gravemeijer et al. 2000; Rasmussen, 1999). As Joaquin varied α, he used these symbolizations dynamically as input for further symbolizations. We take the latter part of his response as characteristic of vertical mathematizing, that creates a model-for reasoning relationally.

More specifically, Joaquin varied the value of α, that he determined "will shift the graph up or down along the d*y*/d*t* axis," as shown in his sketch in Figure 5a. Joaquin then concluded that "Qualitatively, there are five types possible." He then found the specific values of α that result in a bifurcation and made a differentiation between these types. The student symbolized the different types with five different phase lines, as shown in Figure 5b. Joaquin's use of the word "type" is significant for it suggests that for him, each phase line signified all of the solution functions corresponding to each particular value of α. Moreover, each of the five different phase lines is qualitatively different. For example, the first phase line in Figure 5b where $\alpha < 0$ consists of three different regions separated by two equilibrium solutions. The second phase line where $\alpha = 0$ consists of four different regions separated by three equilibrium solutions, and so on. Joaquin then explained how the different phase lines relate to each other and how the equilibria change dynamically as α changes. It is in this sense, as we said earlier, that Joaquin's symbolizing was relational in manner. For example, he stated that "Taking α smaller from graph C (when $0 < \alpha < 1.\overline{185}$), c_1 and c_2 spread apart as b_1 and b_2 approach each other" (where c_1, c_2, b_1, and b_2 all refer to equilibrium solutions). Joaquin's description of two equilibrium solutions spreading apart as α varies and the other two equilibrum solutions approaching each other strongly suggest that his reasoning was based on a dynamic image of the phase line.

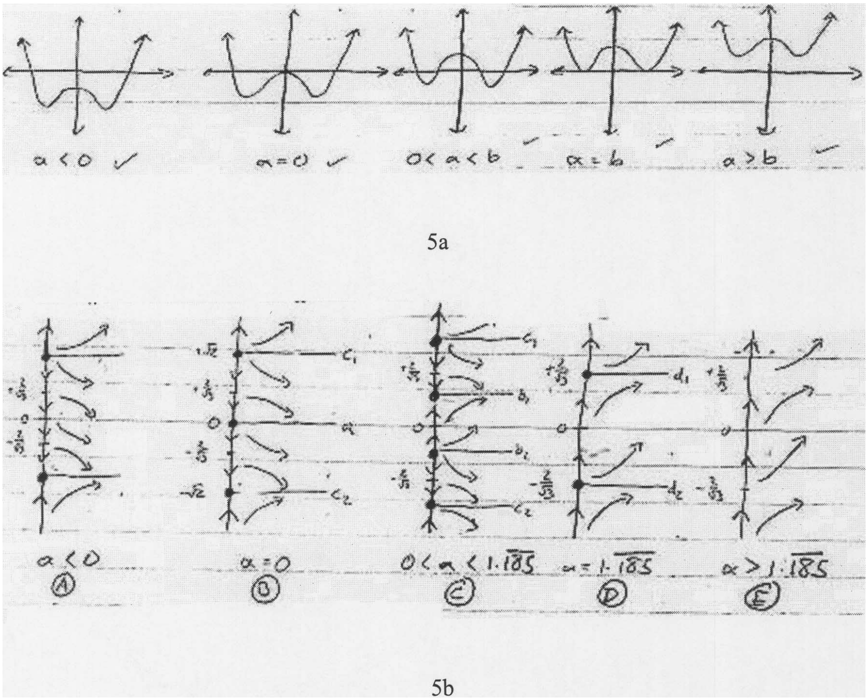


FIGURE 5 Student response on bifurcation task (continued from Figure 4).

Moreover, Joaquin described elsewhere that he had an "epiphany" about how to use the graph of the rate of change equation to figure out the exact bifurcation values. He symbolized, using a sequence of static graphs, a dynamic image of the rate of change equation, as shown in Figure 5a, and he figured out for himself that the key was to determine where the extrema of d*y*/d*t* are tangent to the *y*-axis. Joaquin's use of the word epiphany to describe his reasoning indicates that his solution was not the result of a memorized procedure. It appears that Joaquin had developed a highly integrated and complex way of reasoning about the space of solution functions to differential equations and had developed effective and dynamic symbolizations to foster and further his reasoning.

Taken together, these two examples illustrate how the practice of symbolizing can be viewed as a process of advancing mathematical activity. Indicative of horizontal mathematizing, the phase line was first used as a device to record and communicate student reasoning and conclusions. In the latter part of the second example, this symbol became a tool for reasoning about the generalized space of solution functions in a dynamic manner. Thus, in relation to students' previous activity, this shift represents a form of vertical mathematizing that builds on and extends previous horizontal mathematizing activity. In the following section, a similar progression in activity from horizontal mathematizing to vertical mathematizing is illustrated for the practice of algorithmatizing.

ALGORITHMATIZING

Much of the traditional K–14 mathematics curriculum focuses on students' acquisition of algorithms to do particular types of problems. From double-digit subtraction with regrouping to integration by parts, students' responses across a variety of problems demonstrate that they have acquired these algorithms and can reproduce the appropriate procedures when these procedures are required. As Berlinski (2000) notes, "Algorithms are human artifacts" (p. xvii), the product of human activity. Keeping this activity perspective of algorithms in the forefront suggests that instead of focusing on the acquisition of these algorithms, we can characterize learning to use and understand algorithms as participating in the practice of algorithmatizing. By examining the activity that leads to the creation and use of artifacts, as opposed to the acquisition of the artifacts, we view mathematical learning of and in reference to algorithms through a different lens. This is not to say that the acquisition metaphor (Sfard, 1998) is an unproductive way of viewing the learning and use of algorithms, but rather our development is offered as an opportunity to view algorithmic learning in a different way, that might further enlighten the field. In particular, we examine the use of horizontal and vertical mathematizing within the practice of algorithmatizing.

To illustrate, we summarize an example from the differential equations classroom teaching experiments. In a sequence of three activities, students worked on ways to estimate a solution to a differential equation, leading toward implementing Euler's method (Rasmussen & King, 2000). In the following discussion of horizontal and vertical mathematizing, we use the term "procedure" to indicate steps used to solve a particular task, and the term "algorithm" as a reference for a generalized procedure that is effective across a wide range of tasks.

Algorithmatizing: Horizontal Mathematizing

Instruction prior to the introduction of the following task focused on helping students conceptualize solutions to differential equations as functions, a notion that previous research had identified as needing development (Rasmussen, 2001). Students had also worked to develop a motivation for the reasonableness of a system of differential equations as a model of disease transmission in a closed system. After deriving this system of differential equations, the teacher presented the following task, adapted from Callahan and Hoffman (1995):

> Consider a measles epidemic in a school population of 50,000 children. Suppose that 2,100 people are currently infected and 2,500 have already recovered. Use the following rate of change equations (time measured in days) to estimate the number of susceptible children (S), the number of infected children (I), and the number of recovered children (R) tomorrow and the next day. Organize your data in both tabular and graphical forms.
>
> $$\frac{dS}{dt} = -.00001SI$$
> $$\frac{dI}{dt} = .00001SI - \frac{I}{14}$$
> $$\frac{dR}{dt} = \frac{1}{14}I$$

Students then engaged in a second task, that was grounded in a simpler situation involving fish in a lake. After an initial discussion to establish the reasonableness of the differential equation $\frac{dP}{dt} = kP$ to model unlimited growth of the fish, students engaged in the following task:

> One way to model the growth of fish in a pond is with the differential equation $\frac{dP}{dt} = kP$, with time measured in years. Use this differential equation with a growth parameter $k = 1$ to approximate the number of fish in the pond for the next several years if there are initially (a) 200 fish, (b) 400 fish, and (c) 0 fish. Record your results in tabular and graphical forms.

Students were not supplied with any algorithms with which to approach the two problems. Rather, they had to figure out ways to use the rate of change equations to inform themselves about the quantities of interest. In this way, the mathematical idea of rate of change, together with the specific infectious disease or population scenario, serve as the context or ground from which students proceeded. This approach is in contrast to a traditional practice of first presenting the complete algorithm (Euler's method), with the expectation that students acquire the method and then apply it to a variety of problems. Because of the way in which the tasks were presented, students engaged in the practice of creating procedures for solving particular problems.

Students' efforts reflected their goals and purposes related to predicting future quantities such as the number of infected people or the population of fish. In the service of carrying out these goals, students enacted their understandings of rates of change to do calculations and created tables and graphs to help organize the information to answer questions about population growth or spread of disease. This particularity, the lack of generality of the students' procedures outside of the problem space they were given, is a characteristic of horizontal mathematizing, and forms a basis for a later shift to vertical mathematizing. With rate of change and the context situation as the ground for horizontally mathematizing the problem to develop a procedure, the students had a basis for developing algorithms, that represents a progression or advancement of their mathematical activity.

Algorithmatizing: Vertical Mathematizing

The final task in this sequence asked students to come up with a description (in words and equations) that might help another math or engineering student understand how to approximate the future number of fish in a pond with the differential equation $dP/dt = f(P)$, for some unknown expression $f(P)$. This task, of developing an algorithm, engaged students in the activity of reflecting on and generalizing their previous work. In this case, students began to consider situations in which the time increment need not be one unit and for a variety of types of functions $f(P)$. The procedure needed to be effective across these different situations, and not for a particular differential equation, initial condition, or increment of time. Thus, using their previous activity, students engaged in vertical mathematizing, allowing them to develop generalized formal algorithms.

The practice of developing an algorithm out of several experiences with particular cases represents a vertical mathematizing aspect of algorithmatizing. However, even the above task is set within the particular context of population growth. To move further along the mathematizing continuum, students may be given the following problem:

> Come up with a description (in words and equations) that might help another math or engineering student understand how to approximate a future value of the function $y(t)$ with the differential equation $dy/dt = f(t, y)$.

This task does not refer explicitly to y as a function describing a particular quantity of interest, nor is the differential equation autonomous.[4] Thus the task is intended to foster a further move toward generality.

The algorithm that students develop can then be the ground for further horizontal mathematizing. This happens when, for example, students are presented with particular situations in which they compare an exact solution to the approximate solution generated by their algorithm and they find that their algorithm does not provide reliable long-term predictions. This typically results in students creating explanations for why their algorithm behaves in the way it does in this particular

[4]An autonomous differential equation, dy/dt, is one that depends only on y.

case (horizontal mathematizing). In turn, such activity ultimately leads to their developing a better algorithm useful for all differential equations (vertical mathematizing). As this brief discussion illustrates, the practice of algorithmatizing, like symbolizing and defining, can often consist of more than one layer of horizontal and vertical mathematizing.

The mathematizing progression in this example is paradigmatic of the way in which mathematical activity continues to advance relative to students' previous mathematical activity. Students horizontally mathematized by beginning with a particular problem for which they developed a particular solution procedure. They then developed a generalized algorithm by extending, or vertically mathematizing, their previous activity with this procedure. These algorithms may then become the substance for further horizontal and vertical mathematizing activities. Although this example focuses on collegiate-level mathematics, the same ideas can be useful for elementary and secondary school mathematics. (See Campbell, Rowan, & Suarez, 1998, for an example of algorithmatizing in the early grades.)

DEFINING

Similar to symbolizing and algorithmatizing, the practice of defining can function both as an organizing activity (horizontal mathematizing) and as a means for generalizing, formalizing or creating a new mathematical reality (vertical mathematizing). Creating and using mathematical definitions, versus "everyday definitions," is an essential and often difficult activity for students, even those in upper-level courses such as real analysis (Edwards & Ward, 2004). Freudenthal (1973) distinguishes between two different types of defining activities in mathematics: descriptive and constructive. Descriptive defining "outlines a known object by singling out a few characteristic properties," whereas in constructive defining a person "models new objects out of familiar ones" (p. 457). We use this distinction to help us elaborate horizontal and vertical mathematizing in the domain of defining.

Defining: Horizontal Mathematizing

Descriptive defining is a type of organizing activity (i.e., it is an example of horizontal mathematizing). In a geometry classroom teaching experiment involving undergraduate mathematics, mathematics education, and computer science majors, we asked students to define a number of geometric concepts for which they already had a number of previous experiences in earlier mathematics courses. One of these was triangle. The small group and whole class discussions of possible definitions included arguments over whether the suggested definitions separated examples from nonexamples, whether trivial or extreme examples of triangles should be included, whether a suggested definition was as efficient as it could have been (i.e., did it include redundant characteristics? Should it?), and whether or not the suggested definition included terms that themselves should be defined before their use would be allowed.

These discussions helped clarify what a triangle should be and what criteria were necessary and sufficient to describe such a figure. This type of organizing and clarifying is consistent with what we term as horizontal mathematizing.[5] As we illustrate in the next section, this horizontal mathematizing activity, that in this case can be thought of as descriptive defining, served as the ground or context for activities we characterize as vertical mathematizing.

Defining: Vertical Mathematizing

In contrast to descriptive defining, that singles out characteristic properties of a known object, constructive defining creates new objects by building on and extending these known objects. In the same geometry teaching experiment, we invited students to interpret their planar triangle definition as a definition for triangle on a sphere. Students quickly realized that the only technical change needed was to interpret any mention of straight line in the planar definition as a great circle on the sphere. However, the activity of defining did not stop with this seemingly small change. Students elucidated the definition by creating numerous examples and arguing about whether an example such as the quarter sphere should be called a triangle. (A quarter sphere connects a pole and two antipodal equator points with great circles. Thus, the vertices are colinear, but the area of the figure is positive, unlike when one takes three colinear points on a plane.)

Students struggled to reconcile their planar images of triangle with the planar (now spherical) definition of triangle and the spherical images of possible triangles that they began to generate. This type of activity builds on the previous organizational activity, prompting generalization and abstraction as students use their definition to “define” and create this new concept for themselves through the activity of examining examples. Such defining activity begins to create a new mathematical reality for students—one that consists of new geometric objects and new mathematical relationships between these objects. Such generalizing and abstracting activity that builds on previous mathematizing fits our view of vertical mathematizing. A quote from the journal of a student named Peter captures this experience:

[5]The activity can also be analyzed from another perspective. Students were implicitly developing criteria for what constitutes a “definition.” Thus, they were simultaneously engaged in horizontally mathematizing the process of defining as they created a concept definition specifically for “triangle.” Edwards (1999) elaborates on this notion by analyzing the processes by which students create a concept image (Tall & Vinner, 1981) for mathematical definition.

> From the figure [they had] drawn, it didn't seem like it was a figure at all, but in close observation it was a triangle! Yes, a triangle. It was a triangle based on the definition we chose in class. The definition of a triangle matched up with the figure. Though this was true, the figure did not look like a triangle. I did not see the triangle until someone brought up that it was a triangle by definition. Better yet, there were two triangles! Yes, the inside AND the outside were both triangles.

Students continued generalizing and abstracting as they made conjectures (without prompting from the teacher) about the properties of triangles on the sphere based on the examples that they generated. It is significant that students were generating conjectures about properties of spherical triangles without prompting from the teacher because it suggests that these students were pursuing their goals and purposes in relation to this new mathematical reality of the surface of the sphere. Moreover, this conjecturing process no longer referred back to planar triangles and continued to expand students' notions of spherical triangle, as reflected in another quote from Peter's journal:

> Another surprising observation was when Group 1 gathered information about triangles on a sphere and concluded that the maximum number of degrees that a triangle can have with respect to its angles is 1080 [sic] degrees. These observations [have] changed my view on spheres. All along I was thinking and limited to a 2-dimensional perspective.

As these quotes illustrate, in constructive defining the majority of the elaboration of a concept lies beyond the initiation of the defining activity, beyond the writing or stating of the definition for the first time. In contrast, in descriptive defining the elaboration of the concept occurs primarily before the writing of the definition, hence writing or stating the definition occurs toward the end of the defining activity and the actual agreement on a definition within a certain social structure (e.g., the classroom) is the finishing touch to the defining activity.

The previous example illustrates how students may progress from horizontal to vertical mathematizing by using the organizing activities of horizontal mathematizing as a basis for vertical mathematizing. As illustrated next, such newly formed mathematical realities can become the ground for further mathematizing activity. For example, when investigating whether the condition that two spherical triangles having two sides and the included angle congruent necessarily means that the triangles are congruent, students created a new class of spherical triangles for which this theorem was true. They identified this new mathematical object as a "small triangle" with definitions that varied from group to group. Creating a new class of spherical triangles indicates that triangles on the sphere have become an object in their own right, for students, and the creating of small triangles represents horizontal mathematizing of their new world of spherical triangles. In addition, students considered the equivalence of these various definitions of small triangles and used small triangles as links in chains of deductive reasoning. These mathematizing activities are vertical in nature because they are more formal or abstract in relation to the starting point of defining a triangle on a plane, as well as in relation to their newly created realities of spherical triangles.

As this final example illustrates, advancing mathematical activity can involve more than one layer of horizontal and vertical mathematizing. Students' new mathematical realities (in this case, the world of spherical triangles) that resulted from their earlier mathematizing activities served as the ground for further mathematizing, creating a progressive mathematizing chain or sequence.

CONCLUSION

In the previous examples we developed the idea of advancing mathematical activity by focusing on the nature of students' participation in mathematical practices, elaborating horizontal and vertical mathematizing activities within the practices of symbolizing, algorithmatizing, and defining. Setting horizontal and vertical mathematical activity in relief against each other provides a way to characterize both the nature of students' activity and the progression of this activity. Horizontal and vertical mathematizing activities do not occur in isolation, but comprise a duality referred to as progressive mathematizing. The nature of the activity changes as students shift and slide between what we characterized as horizontal and vertical mathematizing. Horizontal mathematizing capitalizes on students' initial or informal ways of reasoning, with subsequent activities grounded in and building on this work. Participating in the practices of symbolizing, algorithmatizing, and defining facilitates progressive mathematization, generalizations, and the development of new mathematical realities.

It is important to keep in mind that horizontal and vertical mathematizing are of equal value and not intended to reflect some fundamental distinction in the quality or content of cognitive structures (cf. Schwingendorf, Hawks, & Beineke, 1992). Thus, regardless of whether students are learning about compactness or multiplication, the construct of advancing mathematical activity, with its attention to horizontal and vertical mathematizing, is potentially useful for researchers who want to document the development of different types of mathematical practices that emerge in classrooms and for teachers and curriculum developers who want to plan for students' mathematical growth. In particular, advancing mathematical activity via progressive mathematization offers a framework to view two of the three key aspects of what Simon (1995) refers to as a mathematical teaching cycle—the learning goals and a conjectured learning process.

The emphasis on activity, that involves both doing and thinking, resonates with a view of learning as participating in different practices that engage particular goals and purposes of those involved. Essential to the classroom teaching experiments from which we developed the construct of advancing mathematical activity was the fact that explicit attention was paid to explanation and justification. In particular, it became normative for students to routinely explain their thinking in whole class discussions, attempt to make sense of other students' thinking, and indicate agreement or disagreement with other students' mathematical ideas, interpretations, and conclusions. This provided an opportunity to gain insight into the changing nature of mathematical practices, resulting in the development of the notion of advancing mathematical activity.

At the beginning of the article we characterized mathematical learning as participating in mathematical practices. Although not exhaustive, symbolizing, algorithmatizing, and defining were tendered as important examples of such practices. We redirected the discussion on the nature of advanced mathematical thinking to that of advancing mathematical activity. We further put forward the notion of progressive mathematization, composed of horizontal and vertical mathematizing activities, as a means to develop the idea of advancing mathematical activity.

Next, we point to some links and parallels between the practices of symbolizing, algorithmatizing, and defining with respect to horizontal and vertical mathematizing. In the case of all three practices, an important commonality is the interplay between creating and using. The functions that creating and using serve in horizontal mathematizing, however, are different than in vertical mathematizing. In the practice of symbolizing, horizontal mathematizing involved creating a phase line as a record of student reasoning. In the example of algorithmatizing, horizontal mathematizing resulted in creating a procedure to provide future population estimates. In the defining example, creating definitions for planar triangle were an important part of the horizontal mathematizing. Creating the phase line, the procedure, and the definition were done in part to express, support, and communicate ideas that were more or less already familiar, ideas that connected with students' informal or current conceptions.

Further horizontal mathematizing involved using a phase line, using a procedure, and using a definition of triangle. This use, however, remained within the particulars of the problem situation. Using symbols, procedures, and definitions function differently in vertical mathematizing. Using serves the purpose of creating new mathematical realities. The creating in vertical mathematizing is therefore unlike the creating in horizontal mathematizing because, as we said earlier, creating the phase line, the procedure, and the definition were done in part to express, support, and communicate ideas that were more or less already familiar, as opposed to creating new mathematical realities.

Using the phase line, the procedure, and the definition in vertical mathematizing promoted movement from the particular to the more general and in some cases the more formal. In the symbolizing example, students used the phase line to explore the impact of varying a parameter. In the algorithmatizing example, students used their procedure for new and yet-to-be determined differential equations. In the defining example, students used their familiar definition on the not so (mathematically) familiar setting of the sphere. As we illustrated, these uses fostered and promoted the emergence of new mathematical realities for students.

Finally, to what extent might an instructional and curricular focus on advancing mathematical activity help ease what is often seen as a difficult transition "from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions" (Tall, 1992, p. 495)? This transition is indeed difficult when students' intuitive basis founded on experience is an island (Kaput, 1994) separated from their reasoning based on formal definitions and logical deductions. In contrast to a separation of reasoning, the construct of advancing mathematical activity offers teachers, instructional designers, and researchers a practice-oriented way to think about the transition from informal to more formal mathematical reasoning.

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