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# An inquiry-oriented approach to undergraduate mathematics

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**Abstract**

To improve undergraduate mathematics learning, teachers need to recognize and value characteristics of classroom learning environments that contribute to powerful student learning. The broad goal of this special issue is to share such characteristics and the theoretical and empirical grounding for an innovative approach in differential equations called the Inquiry Oriented Differential Equations (IO-DE) project. We use the IO-DE project as a case example of how undergraduate mathematics can build on theoretical and instructional advances initiated at the K-12 level to create and sustain learning environments for powerful student learning at the undergraduate level. In addition to providing an overview of the five articles in this special issue, we highlight the theoretical background for the IO-DE project and provide a summary of two quantitative studies done to assess the effectiveness of the IO-DE project on student learning.


*Keywords:* Inquiry-oriented instruction; Instructional intervention; Realistic mathematics education; Differential equations

A significant number of university mathematics departments, faced with the combination of an increasingly diverse student body, a declining number of mathematics majors, and looming accountability concerns, are more aware than ever of their need to develop effective and innovative curricula and instructional practices (Bok, 2005; Holton, 2001; U.S. Department of Education, 2006). These innovations need to be capable of supporting students in developing deep conceptual understandings of important mathematical ideas as well as productive dispositions. How to accomplish this daunting task is an open question that offers an opportunity for mathematicians and mathematics educators to work together on problems of teaching and learning.

The Inquiry Oriented Differential Equations (IO-DE) project is such a collaborative effort. It seeks to explore the prospects and possibilities for improving undergraduate mathematics education, using differential equations as a case example. In addition to providing an overview of the articles in this issue, the goals of this article are to highlight the theoretical background for the IO-DE project and to provide a summary of two quantitative studies done to assess the effectiveness of the IO-DE project on student learning.

## 1. IO-DE background theory

Perhaps not surprisingly, different research communities characterize inquiry in different ways. For example, in science education the National Research Council (1996) states that inquiry includes identification of assumptions, use of

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critical and logical thinking, and consideration of alternative explanations. In the philosophy of mathematics education, Richards (1991) characterizes inquiry as learning to speak and act mathematically by participating in mathematical discussions, posing conjectures, and solving new or unfamiliar problems. Both characterizations highlight important aspects of student activity. While such characterizations of student activity are essential, they only address part of the process of inquiry. In order to more fully understand the complexity of classroom learning, our definition of inquiry also encompasses *teacher* activity as well as student activity. In particular, IO-DE teachers routinely *inquire* into their students' mathematical thinking and reasoning. Teacher inquiry into student thinking serves three important functions. First, it enables teachers to construct models for how their students interpret and generate mathematical ideas. Second, it provides opportunities for teachers to learn something new about particular mathematical ideas, in light of student thinking. Third, it better positions teachers to build on students' thinking by posing new questions and tasks.

IO-DE students, on the other hand, learn new mathematics through *inquiry* by engaging in mathematical discussions, posing and following up on conjectures, explaining and justifying their thinking, and solving novel problems. Thus, the first function that student inquiry serves is to enable students to learn new mathematics through engagement in genuine argumentation. The second function that student inquiry serves is to empower learners to see themselves as capable of reinventing mathematics and to see mathematics itself as a human activity.

Mathematically, the IO-DE project drew its initial curricular inspiration from contemporary, dynamical systems approaches, such as those developed by Blanchard, Devaney, and Hall (1998) and Hubbard and West (1991). These approaches represent a significant departure from conventional treatments that emphasize a host of analytic techniques for solving special classes of well-posed problems. Stepping away from the conventional approach, Blanchard, Devaney, and Hall, for example, develop analytic, graphical, and numerical approaches as three distinct methods for analyzing solutions to differential equations. They also incorporate application problems to illustrate the usefulness of differential equations in real world problems.

As our extended research team[1] systematically investigated the learning and teaching in such approaches, we developed three goals for inquiry-oriented learning and teaching that extend contemporary dynamical systems approaches.

- First, we wanted students to essentially reinvent many of the key mathematical ideas and methods for analyzing solutions to differential equations. Thus, rather than being introduced to analytic, graphical, and numerical methods as pre-existing, IO-DE students are invited to engage in challenging problems that provide an opportunity for them to create their own analytical, graphical, and numerical approaches. Teachers, for their part, facilitate and support the growth of students' self-generated mathematical ideas and inscriptions, often toward more conventional ones. The three functions of teacher inquiry support this goal.
- Second, and related to the first goal, we wanted challenging tasks, often situated in realistic situations, to serve as the starting point for students' mathematical inquiry. These tasks would stand in stark contrast to application type problems that typically appear at the end of a section. In other words, *experientially real*[2] situations should drive the need for, and creation of, key mathematical ideas that lead to various methods of solving differential equations. Teachers' careful attention to student thinking has helped us identify such experientially real situations.
- Third, we wanted a balanced treatment of analytic, numerical, and graphical approaches, but we wanted these various approaches to emerge more or less simultaneously for learners, rather than as three isolated, disparate methods. The desire for co-emergence of methods stemmed, in part, from earlier research conducted in a reform-oriented differential equations course in which the students exhibited very compartmentalized understandings of analytic, numerical, and graphical methods (Rasmussen, 1997, 2001). For us, a teacher's responsibility to assist students to identify analytical, numerical, and graphical methods as three different techniques comes only *after* the students have made significant progress in reinventing such methods.

[1] The extended research team includes Mark Burtch, Mi-Kyung Ju, Karen Allen Keene, Michael Keynes, Karen King, Oh Nam Kwon, Karen Marrongelle, Bernd Rossa, Wei Ruan, Kyunghee Shin, Michelle Stephan, Joe Wagner, and Erna Yackel.

[2] The term *experientially real* refers to problem situations for which learners can engage their existing ways of reasoning to make progress on the problem. Such experientially real situations may be grounded in real world settings, but depending on the background and experience of learners, may also be grounded in more symbolic, mathematically oriented settings.

Progress in accomplishing these three goals was facilitated through research conducted in three related areas: (1) adaptation of an innovative instructional design approach to the undergraduate level, (2) systematic study of student thinking as they build ideas and identification of teacher knowledge needed to support students' reinvention, and (3) careful attention to the social production of meaning and student identity. Although these three areas have been listed here in a particular order, we hasten to emphasize that they do not represent a linear progression in our research. Indeed, we conduct research in these three areas concurrently and view the research areas as complementary. We now turn to these three research areas of the IO-DE project to situate the articles in this special issue.

### *1.1. Innovative instructional design*

A cornerstone of the IO-DE project is adaptation of the instructional design theory of Realistic Mathematics Education (RME) to the undergraduate level. Central to RME is the design of instructional sequences that challenge learners to organize key subject matter at one level in order to produce new understanding at a higher level (Freudenthal, 1991). In this process, referred to as mathematizing, graphs, algorithms, and definitions become useful tools when students build them through a process of suitably guided reinvention (for illustrative examples and further theoretical development, see Kwon, 2003; Rasmussen, Zandieh, King, & Teppo, 2005).

The mathematization process is embodied in the core heuristics of guided reinvention and emergent models. Guided reinvention speaks to the need to locate instructional starting points that are experientially real to students and that take into account students' current mathematical ways of knowing. The search for such starting points is facilitated by examination of the historical origins of central ideas in differential equations, as well as students' informal solution strategies and interpretations. The heuristic of emergent models highlights the need for instructional sequences to be a connected, long-term series of problems in which students create and elaborate symbolic models of their informal mathematical activity (Gravemeijer, 1999). The term *model* is an overarching idea that refers to student-generated ways of interpreting and organizing their mathematical activity, where activity refers to both mental activity and activity with graphs, equations, etc. From the perspective of RME, there is not just one model, but a series of models through which students first develop *models-of* their mathematical activity leading to *models-for* reasoning about mathematical relationships.

The first article in this issue, by Rasmussen and Blumenfeld, elaborates the emergent model heuristic for student reinvention of solutions to systems of linear differential equations. In contrast to most RME inspired work, particularly at the K-12 level, the authors' analysis extends the construct of emergent models to situations in which symbolic expressions play a prominent role throughout the model-of/model-for transition. In addition, the analysis establishes a connection to the strand on student thinking by highlighting qualitatively different ways that students reason proportionally in relation to the model-of/model-for transition. As such, this article offers a deep analysis of student thinking in relation to our RME inspired instructional design at the undergraduate level.

The second article by Marrongelle illuminates the functions of graphs and gestures in students' reinvention of the Euler method for first order differential equations. It emphasizes how these functions change in students' subsequent use of the Euler method to approximate systems of differential equations. The significance of this article for instructional design, and corresponding teacher support material, is that it offers a lexicon of student gestures along with how these gestures relate to students' reinvention and use of the Euler method algorithm. Such a lexicon could increase a teacher's knowledge about student thinking, and might even suggest ways in which a teacher could intentionally leverage gesturing to support their students' learning.

### *1.2. Research on student thinking and teacher knowledge*

A second cornerstone of the IO-DE project is research that focuses on student thinking and teacher knowledge. For example, research on student cognition in differential equations has highlighted students' concept images of the Euler's method and has examined students' informal or intuitive notions that underlie equilibrium solutions, asymptotical behavior, and stability (Artigue, 1992; Rasmussen, 2001; Zandieh & McDonald, 1999). Knowledge about such informal or intuitive images has been useful for the IO-DE project because it has suggested task situations and instructional interventions that could engage and help reorganize students' informal and intuitive conceptions.

In a case study of a differential equations class that treated contemporary topics in dynamical systems, Rasmussen (2001) found that, rather than building relational understandings (Skemp, 1987), students were learning analytical, graphical, and numerical methods in a compartmentalized manner. An important lesson gleaned from this research is that simply working with multiple modalities will not guarantee that students will build a coherent network of ideas. It is important, in addition, to have a long-term, coherent sequence of tasks. Our adaptation of RME is useful for precisely this purpose.

The third article, by Keene, details student thinking about the use of time as dynamic quantity. In particular, Keene identifies five distinct ways in which students integrate time as a changing quantity as their understanding of systems of differential equations progresses. In contrast to earlier research that indicated time might be an obstacle to student understanding of function (Janvier, 1998), Keene details how time-based reasoning can, in fact, promote and further student understanding of solution functions to systems of differential equations.

Moreover, as our understanding of student thinking evolves, so does our understanding of the kinds of teacher knowledge that would be important for promoting student learning. Beyond content knowledge, such teacher knowledge includes an awareness of students' informal and intuitive ways of reasoning about central ideas in differential equations, knowledge of pedagogical strategies that can connect to student thinking while moving the mathematical agenda forward (Rasmussen & Marrongelle, 2006), knowledge of theoretical ideas related to social aspects of the classroom, as well as mathematical knowledge specific to teaching mathematics in general, and differential equations in particular.

The fourth article by Wagner, Speer and Rossa makes a significant contribution to various types of knowledge that IO-DE teachers find useful for inquiry-oriented teaching. The authors argue that the knowledge required for experienced mathematicians to implement effective, reform practices of instruction in their classrooms includes knowledge that differs from the mathematical content knowledge, pedagogical content knowledge, and pedagogical knowledge that support traditional instruction. Their case study of a mathematician implementing, for the first time, the IO-DE curriculum offers a revealing portrait of the kinds of knowledge essential for inquiry-oriented teaching.

### *1.3. Social production of meaning and identity*

In addition to theoretically informed design with extensive classroom based research, the IO-DE project works from the premise that the way in which instructional tasks are constituted is as important as the material itself. It is toward this aspect that we now turn. An explicit intention of IO-DE project classrooms is to create a learning environment in which students routinely offer explanations of, and justifications for, their reasoning.

Specifically, the constructs of social norms and sociomathematical norms that originated from research in elementary school mathematics classrooms (Yackel & Cobb, 1996) became useful to us in two important ways. First, these constructs offered a way of thinking about the multiple and complementary roles of argumentation as a means to conceptualize processes by which teaching mathematics for understanding can occur (Yackel, Rasmussen, & King, 2000). Second, social norms that empower students to be creators of mathematical ideas, along with the explanations and justifications that support these ideas, provide an opportunity for learners to develop desirable beliefs about the nature of mathematics and their ability to create mathematics (Yackel & Rasmussen, 2002).

In the fifth article of this special issue, Ju and Kwon investigate and document change in students' beliefs about mathematics, about their relation to mathematics, and about their roles in the classroom practice of mathematics. Through their discourse analysis, Ju and Kwon trace a shift from third person perspective to first person perspective as a way to infer changes in students' beliefs. Finally, they point to aspects of the classroom learning environment that are important in transforming students' beliefs, including instructional materials, students' own cognitive resources, and the role of the teacher.

As a whole, the articles in this special issue portray powerful analyses of how inquiry-oriented learning environments can meet the challenges facing today's undergraduate mathematics classrooms. While the qualitative evidence presented by the articles in this issue seems compelling, we also wish to point to two quantitative studies that offer additional evidence that an inquiry-oriented approach to undergraduate mathematics can result in desirable student learning outcomes, as measured by more traditional assessment techniques. We summarize these studies next.

## 2. Quantitative assessments of IO-DE student learning

Rasmussen, Kwon, Allen, Marrongelle, and Burtch (2006) conducted an evaluation study to compare students' routine skills and conceptual understandings of central ideas and analytic methods for solving differential equations between students in inquiry-oriented classes and traditionally taught classes at four different undergraduate

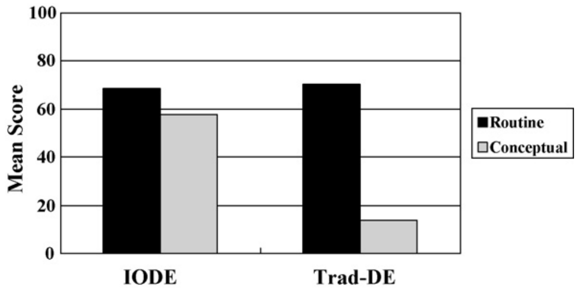


Fig. 1. Mean scores of IO-DE and comparison groups on routine and conceptual tests.

institutions (three in the United States and one in Korea). Whereas IO-DE project classes at all sites typically followed an inquiry-oriented format, comparison classes at all sites typically followed a more traditional lecture-style format.

The assessment consisted of routine skill problems and conceptual understanding problems. Routine skill problems focused primarily on students' procedural fluency with analytic and numerical methods for solving differential equations. On the other hand, conceptual understanding problems aimed at evaluating students' relational understandings of important ideas and concepts. As shown in Fig. 1, there was no significant difference between the two groups on routine problems (IO-DE $n = 65$ and Trad-DE $n = 83$). However, the IO-DE group scored significantly higher than the comparison group on conceptual problems (IO-DE $n = 30$ and Trad-DE $n = 42$). Complete methodological details can be found in Rasmussen et al. (2006).

Further, Kwon, Rasmussen, and Allen (2005) conducted a follow-up study on the retention effect of conceptual and procedural knowledge one year after instruction for a subset of the students from the comparison study. For the purpose of this analysis, procedurally oriented assessment items were defined as those questions that were readily solved via analytic/symbolic techniques. Conceptually oriented assessment items were divided into two categories, modeling tasks and qualitative/graphical tasks, each of which represented important and conceptually demanding thinking in mathematics, in general, and in differential equations, in particular. Each of the two modeling tasks involved determining an appropriate differential equation to fit a given real-world situation. The qualitative/graphical tasks called for predicting and structuring the space of solutions. Fig. 2 shows the post-test and delayed post-test scores of IO-DE and comparison groups on procedurally and conceptually oriented items.

The analysis of these data showed no significant difference in retention between the two groups on the procedural oriented items. However, in retention of conceptual knowledge, as seen from student responses to modeling and qualitative/graphical problems, a significant positive difference was shown in favor of the IO-DE students compared to students in the traditional counterpart.

Based on the results of the post-test and the one year delayed post-test (Kwon et al., 2005; Rasmussen et al., 2006), all IO-DE students from each of the three institutions, regardless of academic backgrounds and gender differences, outperformed traditionally taught comparison students on the post-test. This result demonstrates that the inquiry-oriented instructional approach can be applicable to university mathematics regardless of academic preparations and gender differences.

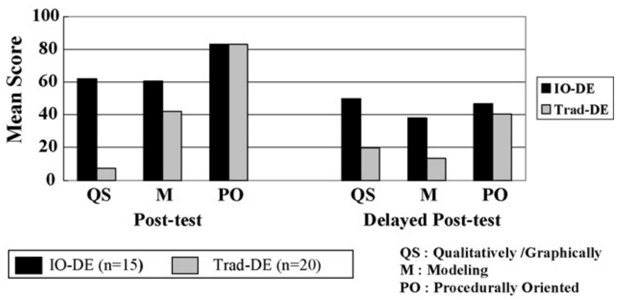


Fig. 2. Students' retention of mathematical knowledge and skills in differential equations.

The articles that form this special issue and the associated theoretical and empirical work that grounds them, suggest that our inquiry-oriented approach can benefit students in undergraduate mathematics classes in several ways. It can help students build the type of conceptual understanding that makes mathematics meaningful to them. It can facilitate students' development of mathematical reasoning ability. It can positively influence their beliefs about knowing and doing mathematics. In light of these features, the IO-DE project may well serve as a model for those interested in exploring the prospects and possibilities of improving undergraduate mathematics education.

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