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Locating starting points in differential equations: a realistic mathematics education approach

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The paper reports on ongoing developmental research efforts to adapt the instructional design perspective of Realistic Mathematics Education (RME) to the learning and teaching of collegiate mathematics, using differential equations as a specific case. This report focuses on the RME design heuristic of guided reinvention as a means to locate a starting point for an instructional sequence for first-order differential equations and highlights the cyclical process instructional design and analysis of student learning. The instance of starting with a rate of change equation as an experientially real mathematical context is taken as a case for illustrating how university students might experience the creation of mathematical ideas. In particular, it is shown how three students came to reason conceptually about rate and in the process, develop their own informal Euler method for approximating solution functions to differential equations.

1. Introduction

The overall purpose of the ongoing work is to explore the feasibility of adapting innovations and theory-driven work at the elementary and secondary level to inform the teaching and learning of university mathematics, using the specific example of differential equations. The focus of this paper is to explicate how the Realistic Mathematics Education (RME) instructional design heuristic of guided reinvention is guiding developmental research efforts. For the past two decades researchers at the Freudenthal institute in The Netherlands have been developing the theoretical perspective of RME to create theoretically and experimentally grounded sequences of instructional activities for the learning of fractions, addition and subtraction, written algorithms, matrices, derivatives, and exponential functions. This paper reports on preliminary efforts to extend this work to post-calculus mathematics.

2. Background

The standard introductory course in differential equations in the USA is currently undergoing substantive change. The traditional course in differential equations, which for the typical engineering or physical science student is a

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second-year course preceded by three semesters of calculus, focuses on various paper and pencil techniques to solve specific types of differential equations. In contrast, several recent reform-oriented textbooks [1, 2] have decreased the emphasis on analytic techniques and have increased the use of computing technology to incorporate graphical and numerical methods for analysing a wide variety of differential equations of real-world concern.

Grounded in evolving interests in dynamical systems and fuelled by reform efforts in calculus, mathematicians are capitalizing on technological advances to incorporate qualitative and numerical approaches that were previously reserved for advanced undergraduate or graduate study in the United States. These changes reflect the increased geometric and qualitative interests of the discipline and provide opportunities for students to learn much more than a collection of specialized techniques for finding exact solutions to differential equations.

Research on the development of students' mathematical reasoning and understanding in differential equations, however, has lagged behind curricular reform efforts. Despite the fact that research in undergraduate mathematics education is a growing area of interest [3–5], very few studies have been completed that examine students' understandings of and difficulties with the concepts and methods of analysis in differential equations (see [6–8] for some exceptions). A basic premise of our work is that mathematics education research can complement reform efforts initiated by mathematicians sensitive to changes in the discipline.

In the sections that follow, we first provide a brief discussion of the theoretical perspective of RME and the project classroom. We then focus on the RME instructional design heuristic of guided reinvention and our ensuing rationale for the starting point in a sequence of instructional tasks intended to develop students' reasoning about approximate solution functions to first-order differential equations. Against this backdrop we then analyse students' reasoning as they engaged in these initial tasks, which in turn serves as rationale for revising and refining the instructional tasks. This cyclical process of design and analysis is characteristic of developmental research.

3. Theoretical perspective

RME is rooted in Freudenthal's interpretation of mathematics as a 'human activity' [9]. From this perspective, students should learn mathematics by mathematizing subject matter from realistic situations (i.e. from context problems or from mathematically real contexts for students) and by mathematizing their own mathematical activity. Mathematizing is characterized by the activities in which one engages for the purposes of generality, certainty, exactness, and brevity [10]. Through a process of progressive mathematization, students should be provided opportunities to reinvent mathematics. Formal mathematics then emerges from students' activities. Central to RME is the role of developmental research. In contrast to traditional instructional design models, developmental research centres on the teaching-learning process, with particular attention to the mental processes of students. Gravemeijer [11, 12] describes how cyclic processes of thought experiments and instructional experiments form the crux of the method of developmental research. He notes that this process is similar to that of the mathematical teaching cycle (MTC) as described by Simon [13]. Analyses in developmental research, such as the one presented here, explicate both the researchers' learning about students' thinking and the pragmatic concerns of revising instructional sequences.

The overall intent for developmental researchers is to constitute a well-considered and empirically grounded instructional theory for the teaching of specific content. In turn, specific instructional theories serve as a means to reconsider and reconstruct the global perspective of RME. In this sense, RME is not a fixed, a priori framework, but a framework that is always under development [12].

4. Project classroom

We conducted a classroom teaching investigation (see [14] for methodological details on classroom teaching investigations) during the Spring 1998 semester at a mid-sized university in the USA with a group of twelve students. As part of the teaching investigation, we collected data including video recordings of all classroom sessions, video-recorded interviews with selected students, copies of students' written work, and records of project meetings.

Guided and informed by our understanding of RME, we developed core learning activities as well as programs for the TI-92 symbolic and graphing calculator. Typically, the instructor briefly introduced these activities; students then worked collaboratively in small groups, which was then followed with a whole class discussion. On occasion, the instructor either used the Interactive Differential Equations (IDE) program [15] during a whole class discussion and/or students used the IDE program in their small group work. For homework assignments and reference purposes, the course used a reform-oriented differential equations text [1] that focused on graphical and numerical techniques to approximate solutions to differential equations and capitalized on the use of technology to treat systems of differential equations prior to second order equations.

5. Discussion

5.1. *Guided reinvention*

A critical aspect of any learning trajectory is the starting point in the sequence of instructional tasks. The principle of guided reinvention suggests two primary ways of locating starting points that are experientially real to students and which take into account students' current mathematical ways of knowing—historical examinations and examinations of students' informal solution strategies and interpretations [12, 16]. Our adaptation of the guided reinvention principle is complemented by consideration of current technological opportunities. It should be noted that the phrase experientially real refers to both realistic contexts and to more abstract mathematical contexts. What is considered experientially real depends on an individual's background and experiences. For example, the mathematical contexts that might be experientially real for a second-year college student are different from those for an elementary school student.

We begin this examination by looking at the historical development of differential equations. The study of differential equations began in the late 17th century with Sir Isaac Newton, who described the *forces* between astronomical bodies and derived the motion from those relationships rather than describing the motions directly [17]. Thus, while other scientists were seeking equations that directly informed them about the motion of the planets, Newton sought information about the motion of planets indirectly, through analysis of rate of change equations. Kuhn [18] likened Newton's work, which challenged fundamental perspectives on gravity, to that of a paradigm shift. Accompanying this paradigmatic shift was a shift in how one might obtain information about a quantity of interest. In particular, Newton's work set the stage for obtaining information about a quantity of interest through its rate of change equation rather than through an equation that directly provided information about the quantity. As such, it took time, thought, and reflection for other scientists to capitalize on Newton's, and parallel work by Leibniz, groundbreaking approach [19]. Similarly, we conjecture that it takes today's students some thought and reflection to deal sensibly with using rate of change equations to obtain information about a quantity of interest. Research also informs us that this shift is not trivial, as evidenced by the difficulty students have in envisioning solutions to a differential equation as collections of functions representing a quantity of interest [7, 8].

In regards to students' informal or intuitive ways of reasoning, examination of the research literature on rate of change and differential equations yielded some insight as well. The work of Thompson and Thompson [20] and Kaput [21] illustrates how pre-college students can reason informally with concepts of rate of change. At the university level, the work of Artigue [6] suggests that when students are taught Euler's method, they have a strong (inappropriate) mental image similar to circles approximated by interior or exterior polygons.

Informed by this initial analysis, we take the mathematical context of instantaneous rate of change equations as an experientially real starting point for students. We situate this mathematical context in broader real world contexts from epidemiology and biology. While students' reasoning with the derivative may not always be mature, we conjecture that it is mathematically sophisticated enough to engage with the tasks presented in this paper in personally meaningful ways. In those cases where this assumption did not hold, we hypothesize that the tasks will help bring to the fore and to stabilize those weaknesses in students' reasoning. We note that this starting point does not literally parallel the historical development, but it might reflect some of the cognitive difficulties associated with the process of using rate of change equations to recover information about a quantity of interest.

5.2. *Instructional tasks*

We presented the tasks in three phases. In the first phase, the instructor described a scenario where a non-fatal, immunity-conferring, communicable disease spreads through a closed community with a fixed population (e.g. chicken pox in a school). He asked the students to sketch the related graphs of the number of susceptible, infected, and recovered populations over time given these assumptions about the disease. By sketching these graphs of the *quantities over time*, the students began to reason about the solutions as continuous functions, as described by their sketches.

For example, one group of students considered the way in which the rate of spread of the disease might affect the solution functions. Their sketch of the graph of the infected population is shown in figure 1. The students in this group said that the actual graph of the infected population would depend on how quickly the disease ran its course, thus all of the graphs depicted were reasonable, whether the progress of the disease was faster or slower.

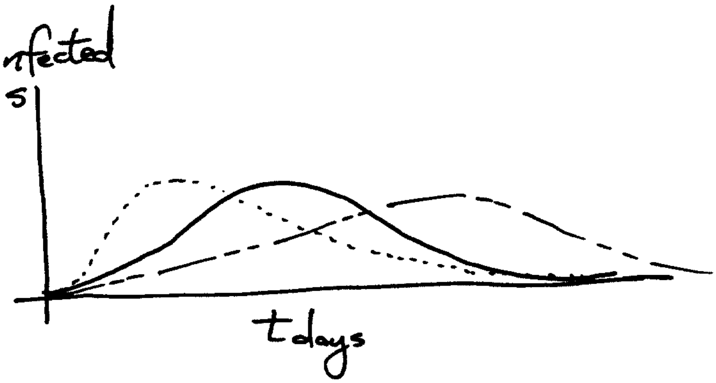

Figure 1. Conjectured infected population graph.

In the second phase the instructor and students had an extensive discussion about how to derive a system of differential equations consistent with the stated assumptions. Upon retrospective analysis, we found that the length of discussion needed to establish the set of differential equations for the virus scenario as viable took longer than we expected and the resultant differential equations themselves did not appear to be well understood by students. In response, we are currently experimenting with a predator–prey scenario in place of the virus scenario.

After derivation of this system of differential equations, the instructor then presented the following task, adapted from Callahan and Hoffman [22].

> Consider a measles epidemic in a school population of 50 000 children. Suppose that 2100 people are currently infected and 2500 have already recovered. Use the following rate of change equations (time measured in days) to estimate the number of susceptible children (S), the number of infected children (I), and the number of recovered children (R) tomorrow and the next day. Organize your data in both tabular and graphical forms.

$$dS/dt = (-0.00001)SI$$

$$dI/dt = (0.00001)SI - I/14$$

$$dR/dt = I/14$$

In contrast to most traditional and reform-oriented approaches where Euler's method is typically presented as a fully formed technique, students were not supplied with a prescribed method or algorithm to approach the problem. Rather, they had to figure out a way to use the rate of change equations to inform them about the quantities of interest. In this sense, the above task was an authentic problem for students and relied on their ability to reason with instantaneous rate of change as a starting point.

Following students' work on this problem and motivated by the need to consider a simpler situation for further analysis, students then engaged in the following task.

> One way to model the growth of fish in a pond is with the differential equation $dP/dt = k^*P(t)$, with time measured in months. Use this differential equation with a growth parameter $k = 1$ to approximate the number of fish in the pond for the next several months if there are initially (a) 200 fish (b) 400 fish (c) 0 fish. Record your results in tabular and graphical forms.

Similar to the previous task, there was an initial discussion to establish the reasonableness of this differential equation to model the growth of fish in a pond. Again, we should note that students still had not received an official algorithm for approaching this problem.

We hypothesized that students might engage with these tasks by maintaining a vision of the solution as a continuous function, which they then try to approximate using the value of dP/dt as an average rate of change over the interval t to $t + \Delta t$. In the next section of this paper we briefly consider alternative ways in which students might have conceptualized the task.

Following these two phases was a critical third phase. In this phase, the final task asked students to come up with a description (in words and symbols) that might help another math or engineering student understand how to approximate the future number of fish in a pond with the differential equation $dP/dt = P(t)$. As we will elaborate in the following paragraphs, this third phase was critical in that it required students to reflect on their previous activity in order to generalize and communicate their reasoning.

5.3. *Analysis*

We begin by briefly discussing students' activity and reasoning on tasks in phase two of the instructional sequence. First, figuring out how to use a rate of change equation to gain information about the quantity itself and how to graphically display this information was not a trivial task for most students. From our perspective this is a positive outcome, as it appeared to promote an atmosphere of genuine problem solving and led to productive student–student and student–instructor interactions. For example, one of the issues that arose for a few students in the first task was whether the amount of change would be the same each day. As a case in point, Gary figured out that the number of children recovered from the virus increased by 150 after the first day of the epidemic and suggested that the number of children recovered the next day would also be 150. Considerable discussion between Gary and his partner ensued before Gary started reasoning recursively with the rate of change equations, as evidenced on later tasks.

Also, in this set of tasks students might immediately recognize the situations where the rate of change is constantly changing and thereby attempt to find a continuous closed form solution function. For example, in the second problem they might immediately attempt to solve the differential equation or use a guess and check method. In fact, at least one group did attempt to use naïve integration methods on this problem. It is possible that they felt that the answer should be some continuous function and attempted to use inappropriate integration methods to find it. However, they were unable to compute a sensible solution and thus had to abandon this method.

Another issue that arose for many students in the second phase of instructional tasks was the manner in which the data points on their graphs should be connected. Should they be connected with straight-line segments or with a smooth curve? For example, Sean and Bill initially connected their data points with a smooth curve, and when asked by the instructor how they decided to draw their graph, they offered the following explanations.

Sean: If it were a straight line then the change would be the same.
Bill: I was thinking that the rate of change is changing so your slope should be changing as you go.

These comments suggest that Sean and Bill are constructing their graphs based on continuous reasoning with the rate of change equations, rather on their discrete computations with the rate of change equations. This example is also evidence that these students were thinking in terms of the goal of the solution function that they were to approximate through their calculations, similar to the graphs of their initial thinking about the situation. Shortly thereafter, Sean and Bill reconsider their graphs based on their calculation, as the following excerpt illustrates.

Bill: We're talking about change in a day. If we interpret breaking down a day into hours and minutes we could curve. If we go by the time being strictly in one day, then you'd go from a straight line to another straight line.
Sean: We have to put in straight lines to be honest.

From this example, we argue that these students recognized the distinction between the instantaneous rate of change in the actual differential equations and the average rate of change that they used in their calculation.

While our position is that Sean and Bill maintained a vision of the solution as a continuous function while using the value of dP/dt as an average rate of change, we consider alternative interpretations as well. For example, some students may have interpreted dP/dt discretely. That is, as $\Delta P/\Delta t$. Thus, instead of approximating a continuous function, they might have conceived of the problem in such a way that they were solving for the actual piecewise linear function with linear interpolation between t and $t + \Delta t$. Alternatively, students might have interpreted the situation in terms of change between fixed integer instants n and $n+1$ where they never considered average or instantaneous rates of change. While we consider this alternative interpretation unlikely due to phase one of the instructional tasks, future researches with students need to carefully consider these alternatives.

Next, we consider the possibility that students failed to make a distinction between rate of change and change. For example, when Bill spoke about 'change in a day' is it possible that he was failing to make a distinction between the rate of change and the change? Certainly for the questions asked in the virus and population tasks there is no numerical difference, only a conceptual difference. Our analysis of students' reasoning while working on the final task, to which we now turn, suggests that this was the case with Bill, but not with Sean. To illuminate this conceptual difference, and to eliminate the possibility that the task fosters the type of reasoning in which Bill engaged, we have since revised the task to consider non-integer time increments. On the day that the instructor handed out this task, a third student, Jerry, chose to join Bill and Sean to work on this problem. We focus our analysis on the work by these three students, since their work most clearly informs our understanding of students' reasoning processes, which in turn aids our pragmatic interests in modifying the instructional tasks.

During the approximately 50 minute time span for which these three students worked on this problem, several issues/dilemmas were either implicitly or explicitly imbedded in these students' discussion. From our perspective, the primary implicit issue/dilemma involved the conceptualization of the situation in a way that involves a rate of change. Secondary issues that became an explicit topic of conversation for these three students included the role and nature of parameters in a differential equation and the role of the units for time.

Above we claimed that conceptualizing the situation in a way that involves a rate was the primary underlying issue/dilemma in these students' work. According to Thompson [23], 'once a situation is conceived in a way that involves a rate, it is implicit in the way the concept of rate is constructed that the values of the compared quantities vary in constant ratio' (p. 192). The data strongly suggests that Sean conceived of the situation in terms of rate, whereas Bill and Jerry initially did not. Our analysis also suggests that through the course of these students' work on this problem, Bill and Jerry eventually conceived of the situation in terms of rate, thus evidencing meaningful mathematical growth. In the following paragraphs, we provide evidence for these claims and illustrate how Bill and Jerry came to conceive of the situation in terms of rate through their interactions with Sean, the instructor, and their constructions with the IDE program and their reflections on these constructions.

The first indication we found that would suggest that Sean conceived of the situation in terms of rate occurred during the third task. While working through the problem with Bill, he said, 'You start out with 400 and 400 times 1 is a change of 400, so it (the population of fish in the pond after 1 month) would be 800.' The fact that Sean said '400 times 1' indicates that he differentiates between rate of change in the population and change in the population. Through the course of these students' work on the final problem, there were numerous other instances that substantiate our claim that Sean conceived of the situation as rate. The excerpt below is exemplar of the kind of explanations Sean was offering to both the instructor and his partners.

Sean: Like if your (time increment is) 1 month, and you find out that your change in 1 month is 100, you take that and divide it in half, so you go 1/2 month so you want 1/2 change. So now you go 200 (the initial population) plus 50. Now you're at 250. You then figure from that 1/2 month on up, from 250.

Although we have demonstrated that Sean conceives of the situation as rate, it is not clear from the available data if Sean is explicitly aware of the possibility that Bill and Jerry do not initially make the distinction between rate of change and change. In fact, many of Sean's explanations were in terms of procedural instructions, as in the excerpt above, and seemed to have little impact of helping Bill and Jerry re-conceptualize the situation in terms of rate.

In contrast, our claim is that Bill and Jerry did not initially conceive of the situation as rate, and therefore did not make a distinction between rate of change and change. For example, when Sean, Bill and Jerry begin work on the third task, Bill creates the following symbolic account intended to describe their approximation process: $P_{(\text{initial})} + \mathrm{d}P/\mathrm{d}t \rightarrow P_{(\text{initial})}$ [1](sic). Jerry summarizes this by saying,

[1] The students perhaps conceived the equation in a way consistent with computer programming in which one may use the variable to redefine itself.

'You take the current population and add to it the change relative to time to establish the new population, then continue the process.' Notice that in both Bill's symbolic description and Jerry's verbal description there is no distinction between rate of change and change. This bit of information alone, however, is insufficient to conclude that Bill and Jerry did not conceive of the situation in terms of rate. The fact that the previous tasks were framed in terms of either 1 day or 1 month time intervals might be masking the manner in which they conceive of the situation. Based on what happens next, however, this does not appear to be a viable interpretation.

At this point, the instructor joins their small group, listens to their explanation of their results, asks them how they might change their work if they wanted to approximate the number of fish in the pond in 1/2 month increments instead of in monthly increments, and then leaves. If Bill and Jerry conceived of the situation in terms of rate, this would be a fairly straightforward question to resolve. However, it is not until approximately 45 minutes and considerable discussion later that Bill, Jerry and Sean resolved this dilemma and created the following symbolic description:

$$P_{(initial)} + \left(\mathrm{X}\frac{\mathrm{d}P}{\mathrm{d}t}\right) \begin{matrix}\textit{Fraction} \\ \textit{of the month}\end{matrix} \rightarrow P_{(new)}$$

How was it that Bill and Jerry eventually came to conceive of the situation as rate? Their initial thought was that the parameter k would have to change each time in order for the approximation with 1/2 month time increments to yield the same approximation as with the monthly time increments. This might reflect an implicit mental image of their approximations similar to inscribed polygons in a circle. Or, they may have been attempting to approximate their approximation, in a way of reasoning similar to that previously described, in which the discrete, piecewise linear approximation is the 'solution'.

Upon rejoining the group the instructor used their initial thinking about changing k as an opportunity to explain the nature of parameters in a differential equation. He then suggested that they experiment with the slope field tool in the IDE program. With this tool, students first select the differential equation of interest (in this case $\mathrm{d}x/\mathrm{d}t = x$) and then they can click the mouse anywhere in the (t, x) plane to produce a segment of the slope vector with horizontal displacement equal to the size of the time step. The user can also visually compare slope vectors for different time steps.

After experimenting with this program for some time, there was a gradual shift in the way in which Bill and Jerry were reasoning. The sketch in figure 2 provides a rough idea of the type of screen displays they created as they experimented with this program, displaying slope approximations with the same initial population, but with $\Delta t = 1$ and $\Delta t = 0.5$.

An important occasion for these students' learning came while interacting with Sean and on occasion the instructor and reflecting on their activity with the IDE program. In particular, as expressed by Jim, they figured out that,

> the first vector is always the change in population over the change in time, regardless of the length of that vector. It's always going to have the same slope. And that's why the longer the time period the flatter the curve is going to be.

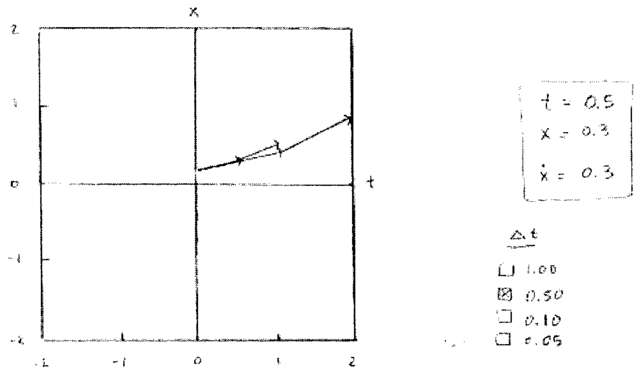

Figure 2. Hand sketch of IDE screen display.

Further evidence that Jim is beginning to reason in terms of rate comes a few minutes later when they are trying to figure out how to express their thinking symbolically, In response to Sean's comment that, '$\mathrm{d}P$ changes at the same proportional rate as $\mathrm{d}t$. When you have $x\,\mathrm{d}P$ you must also have $x\,\mathrm{d}t$,' Jerry says,

> If you look at the calculation, all you're doing is getting an improved approximation because the growth rate is kP, period ... So when you take 1/12th of the time, you take 1/12th of the population so that you keep the slope the same.

Although Bill was less verbal than Jerry, we interpret Bill's frequent nodding, lack of objections or questions, and writing up the groups' response to indicate that he also made some shifts and is conceptualizing the situation in terms of rates. Note that the image associated with Jim's reasoning does not reflect an image of a polygon inscribed in a circle. Indeed, as judged by subsequent tasks, Jerry, Bill and Sean did not appear to have a mental image of Euler's method like that of circles approximated by interior or exterior polygons.

As discussed above, these three students' activities with rate of change equations promoted their conceptual understanding of rate and provided opportunities for them to experience the reinvention of mathematical ideas. In particular, these students developed their own informal Euler's method. More conventional or formal ways of expressing Euler's method were then able to emerge out of and be grounded in students' informal activity.

6. Concluding remarks

Regarding limitations and modifications of our approach, we found that the length of discussion needed to establish the set of differential equations for the virus scenario as viable took longer than we expected and the resultant differential equations themselves did not appear to be well understood by students. In response, we are currently experimenting with a predator–prey scenario in place of the virus scenario. More importantly, the analysis reported here strongly suggests to us that we include a variety of non-unit time increments in the tasks. We suspect that this will bring the issue/dilemma regarding students' conception of the situation in terms of rate to the fore much sooner than it did in this classroom teaching investigation. It might also bring forward the meaning of numerical approximation and the relationship between the discrete approximation and the continuous solution.

Our analysis of these three students' learning also suggests that the IDE program is a promising means for promoting the kind of conceptual reasoning about rate that we wanted to foster. As is probably the case with other universities, we don't always have access to classrooms equipped with the necessary computers. As such, we are currently exploring means by which the kind of learning opportunities that arose for these three students while using the IDE program might also be part of programs for the TI-92 graphing and symbolic calculator.

Besides contributing to the normative understanding that mathematics is a sense making activity, students' reinvention of Euler's method also serves as a departure point for further mathematical development. From the instructional design perspective of RME, instructional tasks should be experientially real for students and be justifiable in terms of potential endpoints. Toward this end, we are investigating pedagogical and technological means by which the slope field might emerge as an initial record of students' reasoning and mathematical activity with their numerical approximation. A complete discussion of this aspect of the project, however, is beyond the scope of this paper.

The broader significance of this work will be the extent to which the perspectives and approaches we develop can inform and guide other university instructors as they strive to promote mathematical growth with their students in their classes. Current reform efforts in differential equations include many thoughtful and innovative treatments of the mathematical content. The approach being developed here seeks to build on and complement these positive aspects by adapting principled approaches and perspectives that have informed the re-thinking of mathematics learning and teaching at the elementary and secondary level to the re-thinking of mathematics learning and teaching at the university level.

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References

[1] Blanchard, P., Devaney, R., and Hall, R., 1998, *Differential Equations* (Boston: Brooks/Cole).

[2] Kostelich, E., and Armbruster, D., 1997, *Introductory Differential Equations: From Linearity to Chaos* (Reading, MA: Addison-Wesley).

[3] Dubinsky, E., Schoenfeld, A. H., and Kaput, J. (eds), 1994, *Issues in Mathematics Education: Vol. 4. Research in Collegiate Mathematics Education. I* (Washington, DC: American Mathematical Society & Mathematical Association of America).

[4] Kaput, J., Schoenfeld, A. H., and Dubinsky, E. (eds), 1996, *Issues in Mathematics Education: Vol. 6. Research in Collegiate Mathematics Education. II* (Washington, DC: American Mathematical Society & Mathematical Association of America).

[5] Schoenfeld, A. H., Kaput, J., and Dubinsky, E. (eds), 1998, *Issues in Mathematics Education: Vol. 7. Research in Collegiate Mathematics Education. III* (Washington, DC: American Mathematical Society & Mathematical Association of America).

[6] Artigue, M., 1992, in *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by G. Harel and E. Dubinsky (Washington, DC: Mathematical Association of America), pp. 109–132.
[7] Rasmussen, C., 1998, Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.
[8] Zandieh, M., and McDonald, M., 1999, Paper presented at PME-NA XXI, Cuernavaca, Mexico.
[9] Freudenthal, H., 1991, *Revisiting Mathematics Education* (Dordrecht: Kluwer Academic).
[10] Gravemeijer, K., Cobb, P., Bowers, J., and Whitenack, J., in press, in *Symbolizing, Modeling, and Instructional Design*, edited by P. Cobb, E. Yackel, and K. McClain.
[11] Gravemeijer, K., 1998, Paper presented at the Research Pre-session of the Annual Meeting of the National Council of Teachers of Mathematics, Washington, DC.
[12] Gravemeijer, K., 1998, in *Mathematics Education as a Research Domain: A Search for Identity*, edited by A. Sierpenska and J. Kilpatrick (Dordrecht: Kluwer Academic), pp. 277–296.
[13] Simon, M., 1995, *J. Res. Math. Educ.*, **26**, 114–145.
[14] Cobb, P., 2000, *Handbook of Research Design in Mathematics and Science Education*, edited by A. E. Kelly and R. Lesh (Mahwah, New Jersey: Lawrence Erlbaum & Associates), pp. 307–334.
[15] West, B., Strogatz, S., McDill J. M., and Cantwell, J., 1997, *Interactive Differential Equations CD-ROM* (New York: Addison-Wesley Interactive).
[16] Streefland, L., 1991, *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research* (Dordrecht: Kluwer Academic).
[17] Hubbard, and West, B., 1991, *Differential Equations: A Dynamical Systems Approach, Part 1* (New York: Springer-Verlag).
[18] Kuhn, T., 1990, *The Structure of Scientific Revolutions*, 2nd edn (Chicago: The University of Chicago Press).
[19] Kline, M., 1972, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press).
[20] Thompson, P. W., and Thompson, A. G., 1994, *J. Res. Math. Educ.*, **25**, 279–303.
[21] Kaput, J. J., 1994, *Mathematical Thinking and Problem Solving*, edited by A. Schoenfeld (Hillsdale, NJ: Lawrence Erlbaum), pp. 77–156.
[22] Callahan, J., and Hoffman, K., 1995, *Calculus in Context* (New York: W. H. Freeman).
[23] Thompson, P., 1994, *The Development of Multiplicative Reasoning in the Learning of Mathematics*, edited by G. Harel and J. Confrey (Albany: State University of New York Press), pp. 179–234.