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Analyzing student understanding in linear algebra through mathematical activity

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ABSTRACT

In this paper we characterize students' conceptions of span and linear (in)dependence and their mathematical activity to provide insight into their understanding. The data under consideration are portions of individual interviews with linear algebra students. Grounded analysis revealed a wide range of student conceptions of span and linear (in)dependence. The authors organized these conceptions into four categories: travel, geometric, vector algebraic, and matrix algebraic. To further illuminate participants' conceptions of span and linear (in)dependence, the authors developed a categorization to classify the participants' engagement into five types of mathematical activity: defining, proving, relating, example generating, and problem solving. Coordination of these two categorizations provides a framework that proves useful in providing finer-grained analyses of students' conceptions and the potential value and/or limitations of such conceptions in certain contexts.

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1. Introduction

The purpose of the study reported in this paper was to investigate student understanding of span and linear (in)dependence in linear algebra and to contribute to the body of knowledge regarding how individuals understand undergraduate mathematics. This study fits within a larger research program in which we explore students' transitions from informal to more formal ways of reasoning in linear algebra and leverage that research to produce curricular materials that promote a student-centered, inquiry-oriented approach to the teaching and learning of linear algebra. In particular, our research goals for the current study were (a) to classify students' conceptions of span and linear (in)dependence, and (b) to investigate how students use these conceptions to reason about relationships between span and linear (in)dependence. We first oriented our analysis of data from individual interviews through a grounded theory approach (Glaser & Strauss, 1967) in order to identify student conceptions of span and linear (in)dependence. We noticed that in coding students' conceptions, for which we made use of Tall and Vinner's (1981) construct of concept image, our analysis was facilitated by noting the type of mathematical activity in which the students were engaged as they shared their ways of reasoning. In other words, the interview question to which a student was responding had the potential of eliciting different aspects of the student's concept images. This is consistent with Vinner's (1991) notion of *evoked concept image*. For example, a student's reasons why a claim was true or false revealed ways of thinking about the associated concepts differently than did his or her response to "how do you personally think about this concept?" As such, we identified within the data set five mathematical activities in which students engaged during the interviews: defining, proving, relating, example generating, and problem solving. Within

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this paper we show how these mathematical activities can be used as a lens to further refine characterizations of students' concept images of span and linear (in)dependence.

Given these activity categories, our refined research objectives are (a) to investigate students' concept images of span, linear (in)dependence, and relationships between the two concepts and (b) to use the mathematical activities of defining, proving, relating, example generating, and problem solving in coordination with students' concept images in order to provide deeper insight into their understanding. Section 5 details the four concept image categories that grew out of our data: travel, geometric, vector algebraic, and matrix algebraic. We also define the five mathematical activities and provide examples of how coordination of the concept image categories with mathematical activity categories informed analysis of student thinking. Finally, we detail a framework of dual categorizations, provide an example using one student's response, and use this to provide richer descriptions of three students' understanding of linear independence, linear dependence, and span.

2. Theoretical perspective and literature review

The larger research program from which these data are drawn is framed by Cobb and Yackel's (1996) emergent perspective. From an assumption that that mathematical development is a process of active individual construction and a process of mathematical enculturation, this framework coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). Because the current research focuses on individual students' understanding within an interview setting, we restrict our analysis to the mathematical conceptions that individuals bring to bear in their mathematical work (Rasmussen, Wawro, & Zandieh, 2015). Within our analysis, we are guided by the assumptions that learners acquire knowledge from their daily experiences, that prior conceptions affect interaction with new ideas, and that knowledge structures are contextually dependent (diSessa, 1993). As such, we do not claim that our analyses of students' responses are the exact way that the participants thought about the concepts at the time of the interview; rather, we view their communication with the interviewer as data that acts as a proxy for how they think and reason about the mathematical content. This orientation to research aligns well with the use of Tall and Vinner's (1981) concept image framework, which facilitates our characterization of the nuanced ways in which individuals conceptualize mathematical ideas.

Given that the importance of linear algebra in the undergraduate mathematics curriculum because of both its wide applicability in the sciences and its pivotal role in the transition into more abstract and formal mathematics (Harel, 1989), the body of research regarding the teaching and learning of linear algebra has grown over the past few decades. The volume, *The Teaching and Learning of Linear Algebra*, edited by Dorier (2000), rises to the fore as a particularly influential collection of results in this research area. This volume includes empirical research regarding the "object of formalism" and teaching interventions that take this into account (Dorier, Robert, Robinet, & Rogalski, 2000; Rogalski, 2000), as well as various categorizations for modes of thinking and description in linear algebra (Hillel, 2000; Sierpinska, 2000). This latter work by Hillel (2000) and Sierpinska (2000) is of particular interest for this paper on analyzing student understanding of span and linear independence.

Hillel suggested three possible modes of description for vectors and vector operations, namely geometric, algebraic, and abstract. The *abstract mode* uses language of generalized theory, including terms such as dimension, span, linear combination, and subspace. The *algebraic mode* uses concepts more particular to the vector space $\mathbb{R}^n$, such as matrix, rank, and systems of linear equations. Finally, the *geometric mode* uses language that is familiar from our lived experiences, such as point, line, plane, and geometric transformation (p. 192). Hillel details difficulties students have within a given mode (such as confusion potentially caused by describing vectors as both arrows and points, both of which are a geometric description of vectors), as well as in moving between modes (such as how the difficulty in change of basis problems within $\mathbb{R}^n$ may relate to switching between algebraic and abstract modes). Connecting to this work, Larson (2010) noted that students often seem to blend these modes of representation. For instance, she gives examples such as "span of a matrix" or "linearly dependent matrix." These may arise from a blending of different modes (namely abstract and algebraic). This might also arise from students misattributing properties of a set of vectors, such as span and linear independence, to a matrix, although students may speak metonymically (Lakoff & Johnson, 1980) as if a single matrix is itself the set of vectors.

Attributing them to the historical development of linear algebra, Sierpinska (2000) suggests three modes of thinking and reasoning that coexist in linear algebra: synthetic-geometric, analytic-arithmetic, and analytic-structural. The first mode focuses on spatial reasoning, the second on algebraic manipulation and representation, and the third on formal, theorem-based and axiomatic thinking. The work of Dogan-Dunlap (2010) uses this framework to characterize students' descriptions of linear independence and dependence. By comparing student responses to a written assignment with one including a geometric component via an online dynamic graphing module, Dogan-Dunlap found 17 different categories of student thinking. The author further labeled the categories of student responses as either geometric or algebraic/arithmetic (the author uses "algebraic" interchangeably with "structural"), determining that 11 categories from across multiple question responses could be labeled as geometric. Dogan-Dunlap concluded that the online dynamic module facilitated students' development of geometric thinking and integration of multiple modes of thinking, noting, "the geometric representations in the presence of algebraic and arithmetic modes appear to help learners begin to consider the different representational aspects of a concept" (p. 2158).

A large portion of the research on student learning of span and linear independence in linear algebra (Aydin, 2014; Bogomolny, 2007; Ertekin, Solak, & Yazici, 2010; Kú, Oktaç, & Trigueros, 2011; Stewart & Thomas, 2010; Trigueros & Possani,

2013) is aligned with Action-Process-Object-Schema (APOS) theory. APOS theory (Dubinsky & McDonald, 2001) has its roots in Piagetian constructivism and aims to delineate various stages of mental construction through which learners progress as they develop understanding of advanced mathematical ideas. For instance, Kú et al. (2011) detail a genetic decomposition – which is an idealized description of the actions, processes, objects, schemas that develop as a learner constructs an understanding of a mathematical concept – for the notions of spanning set and span. To summarize, their genetic decomposition posits that students display: (a) an action conception of spanning set if they show a process conception of linear combination and if they verify that a set spans a certain vector space with only specific vectors; (b) a process conception of spanning set if they generalize the previous actions to realize that every vector in a vector space can be written as a linear combination of the vectors in the spanning set; (c) an object conception if they can apply actions to the spanning set and describe its defining attributes. When tested against empirical data, Kú et al. determined that the concept of spanning set relied on a process conception of both set containment and linear combination, and – of particular relevance to this paper – that students who displayed at least a process conception of spanning set were able to relate that to other fundamental linear algebra concepts such as linear independence.

With respect to linear independence, researchers' genetic decompositions of the topics are varied. For instance, both Aydin (2014) and Bogomolny (2007) provide genetic decompositions of linear independence based on students' work of generating 3×3 matrices with linearly dependent rows or columns. Both works typified an action conception as a focus on single entries and matrix operations such as row-reduction, and an object conception as a focus on linear combinations of rows or columns. Trigueros and Possani (2013) posit that a genetic decomposition for linear independence builds from three of students' prior schema – algebraic, vector, and set – toward students' development of an object conception of linear combination, which is central in a linear (in)dependence schema. Trigueros and Possani used this work as a foundation from which to create a modeling task sequence designed to foster students' development of an object view of linear independence.

Finally, Stewart and Thomas (2010) coordinated APOS Theory (Dubinsky & McDonald, 2001) with Tall's (2004) Three Worlds of Mathematics to characterize the various possibilities for student understanding of linear independence, span, and basis according to the authors' genetic decomposition of the concepts. The authors described possible action, process, and object conceptions, as well as possible characterizations according to the three worlds of mathematics – Embodied, Symbolic, and Formal – in a coordinated way for each linear algebra topic. They analyzed students' responses (from two different classes) using to this dual framework and found that students in the more traditional course relied heavily on matrix manipulation (classified as a process-symbolic matrix conceptions) with little connection being drawn between the matrix manipulation and the associated concepts. Also, the participants in the study seemed to have limited understanding of linear combination, upon which the concepts of span and linear (in)dependence rest. The authors recommended instruction that focuses on geometrically grounded development of the concepts (specifically linear combination), cautioned against too much reliance on the embodied aspects of linear algebra, and suggested that an appropriate balance would develop the concepts from a more geometric approach and relate these concepts at a more formal level.

The aforementioned literature has informed and shaped the research we present in this paper. We drew inspiration with respect to the various modes of description that are possible within linear algebra, such as geometric and algebraic, as aspects of student reasoning to be sensitive to within our analysis. We also drew inspiration as we highlighted what objects, namely vectors or matrices, students reasoned with as they explained their thinking about span and linear (in)dependence. While these studies expand our knowledge of student conceptions of span and linear (in)dependence, our analysis of student conceptions are grounded with no a priori categorizations. As such, we also draw from the notion of concept image (Tall & Vinner, 1991), which has been adopted as an analytical tool for characterizing students' conceptions of ideas within linear algebra. This approach allows categorizations to surface from the data through grounded theory, rather than through a priori classifications. For example, Zandieh, Ellis, and Rasmussen (2012) investigated students' conceptions of function and linear transformation and discovered five concept image categories for linear transformation within their data, such as morphing (an input morphs into an output) or machine (a transformation acts on an input to produce an output). Wawro, Sweeney, and Rabin (2011) documented concept image categories for students' conceptions of subspace within linear algebra, namely part of whole, geometric object, and algebraic object. Although these categories were grounded in data, there exist points of compatibility with Hillel's and Sierpinska's groupings.

3. Setting and participants

The data for this study come from a semester-long classroom teaching experiment (Cobb, 2000) conducted in an introductory linear algebra course at a large public university. Classroom instruction was guided by the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991), with the goal of creating a linear algebra course that built on student concepts and reasoning as the starting point from which more complex and formal reasoning developed. The class engaged in various RME-inspired instructional sequences focused on developing a deep understanding of key concepts such as span and linear independence (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012), linear transformations (Wawro, Larson, Zandieh, & Rasmussen, 2012), Eigen theory, and change of basis. The instructional sequences consist of a set of experientially real tasks that allow for active student engagement in the guided reinvention of key mathematical ideas. On one level, students learn mathematics though inquiry by participating in mathematical discussions, explaining their thinking, and solving novel problems. On a second level, instructors inquire into their students' mathematical thinking to make decisions and guide classroom activity (Rasmussen & Kwon, 2007). As such, the instructor structured class time to allow

"Suppose you have a 3 x 3 matrix A, and you know that the columns of A span $\mathbb{R}^3$. Decide if the following statements are true or false, and explain your answer:"

Question 1a

The column vectors of A are linearly dependent.

Follow-ups. Skip if redundant:

- "How do you think about span?"
- "How do you think about what it means for vectors to be linearly dependent?
- "How does linear dependence relate to span of a set of vectors?

Question 1b

The row-reduced echelon form of A has three pivots.

Follow-ups. Skip if redundant:

- "How do you think about what a pivot is?
- "How do pivots of a matrix relate to span of a set of vectors?"

Fig. 1. Interview questions analyzed for this study.

for small group work, whole class discussion, partner talk, and to align the students' mathematical activity with that of the broader mathematics community. The textbook *Linear Algebra and its Applications* (Lay, 2003) was used as a supplemental resource.

The five participants in this research (pseudonyms) – Abraham (a junior statistics major), Giovanni (a senior business major), Justin (a sophomore mathematics major), Aziz (a junior chemical physics major), and Kaemon (a senior computer engineering major) – participated in semi-structured individual interviews (Bernard, 1988) the week after final exams. Although we only provide a detailed discussion of our analysis of three students' responses in this article (Abraham, Aziz, and Kaemon), data from all five students contributed to the development of the two categorizations. Each interview lasted approximately ninety minutes. The purpose of the interview was to investigate how students reasoned about the concept statements that comprise the Invertible Matrix Theorem; the entirety of the interview protocol can be found in Wawro (2011). Questions 1a and 1b (see Fig. 1) of the protocol were used to elicit students' ways of thinking and reasoning about span, linear (in)dependence, and how they relate to each other. "The columns of A span $\mathbb{R}^n$" and "the columns of A are linearly independent" are equivalent statements for any $n \times n$ matrix A. By beginning the interview with a true/false question about a 3×3 matrix A, paired with open-ended follow-up questions about students' conceptions about span and linear independence in general, we hoped to elicit responses comprised of rich descriptions, visual imagery, and connections between concepts. The interview questions analyzed in this study are given in Fig. 1. Video recordings and transcripts of the interviews served as primary data sources, with all written work serving a secondary role.

4. Methods

Videos and transcriptions of the participants' responses to Questions 1a and 1b (Fig. 1) were iteratively analyzed. In the first analysis, we focused on the logical progression of the participants' argumentation and what mathematical objects the participants attributed as 'acting' in different parts of their discussion (i.e., "the matrix spans $\mathbb{R}^3$" or "the vector moves in this direction"). A summarizing process that described the participants' general progression followed this analysis. A second analysis parsed out students' conceptions of linear (in)dependence and span, and we oriented our analysis through a grounded theory approach (Glaser & Strauss, 1967). We also separated general discussion of a concept from instances in which the interviewer directly asked the student to define the given concept. Quotes were drawn from the transcript and grouped by which concept the student was arguing with or describing. Two participants' quotes were categorized and iteratively compared, forming open codes, which were in turn developed into focused codes used with the remaining participants' data. It was during this coding process that distinctions between types of activity became clear and led to the emergence of the five categories of activity. In the next iteration of analysis, we categorized student quotes according to these five emerging activities and separated the quotes according to span, linear independence, or linear dependence. These quote collections were then compared for categorical similarities and differences. At every stage of this process, we continually questioned and challenged each other's decisions, such as motivation for choice of categorization or interpretation of a student's quote (Denzin, 1978).

5. Results

The participants in this study used a variety of language to describe their conceptions of span, linear (in)dependence, and how they relate to each other. We organized this variety into four concept image categories: travel, geometric, vector algebraic, and matrix algebraic (see Table 1). We also identified five mathematical activities in which students engaged during the interview: defining, proving, relating, example generating, and problem solving (Table 2). Within this section, we first detail the concept image categories and the mathematical activities. We then illustrate the dual coding with an example from Justin's interview. The remainder of the section uses responses from three of the participants to illustrate how

Table 1
Summary of the concept image categories for span and linear (in)dependence.

Category	Description
Travel	• Language indicative of purposeful movement • Captures notions of "getting to" or "moving to" locations in the vector space
Geometric	• Language indicative of spatial reasoning or graphical representations without use of travel-oriented language • Included sketches of vectors and/or discussion of objects such as lines and planes
Vector Algebraic	• Participants use operations on algebraic representations of vectors in description • Includes linear combination of vectors written as $n \times 1$ matrices or designated by variables (i.e., $2\mathbf{v}+3\mathbf{w}$)
Matrix Algebraic	• Involves explicit attention to the form or properties of a matrix (e.g., size, actual values, pivots) • Participants focus on operations on matrices (e.g., Gaussian elimination)

Table 2
Summary of the mathematical activity categories.

Mathematical Activity	Definition
Defining	The act of describing a concept's essential qualities
Proving	The act of providing a justification to a claim
Relating	The act of comparing, contrasting, or explaining connections between different concepts or between different interpretations of the same concept
Example generating	The act of creating cases of certain concepts or properties (e.g., a set of three linearly dependent vectors in $\mathbb{R}^3$)
Problem solving	The act of engaging in a calculation, algorithm, or reasoning with a specific goal to determine a previously unknown result

the coordination of the concept image categories and activities lends itself to a framework that provides insight and nuance to characterizations of students' ways of reasoning about span and linear (in)dependence.

5.1. Categories of student conceptions

The *travel* category captures students' description of span and linear (in)dependence in terms indicative of purposeful movement. While this category is consistent with spatial and geometric reasoning, it is more specific in that it captures notions of "getting to" or "moving to" locations in the vector space under consideration. The participants' travel conceptions of span were indicated by phrases such as "everywhere you can get" (stated by Justin when describing the span of a set of vectors) or "the vectors can take you anywhere [in $\mathbb{R}^3$]" (stated by Giovanni when describing what it means for vectors to span $\mathbb{R}^3$). With respect to linear independence, participants' travel conceptions included phrases such as "[the vectors] only move farther away" (Justin). A travel conception of linear dependence was generally indicated by phrases such as "then that would make that linearly dependent because I can, I can kind of get there and take that vector back" (Abraham), and "you can move one way on one vector, second way, and then take the third one back to the origin" (Aziz). Participants often negated phrases they used to describe sets of linearly independent vectors.

The *geometric* category is used to capture language indicative of spatial reasoning or graphical representations without use of travel-oriented language. This includes student explanations in which vectors are represented graphically on 2- or 3-dimensional axes, or when explanations include sketches or mention of objects such as lines, planes, or areas. For instance, when asked to discuss the span of three linearly dependent vectors he had given as an example, Kaemon stated, "So since it's, like you're on this one line, you can't really get all the combinations that are in these quadrants." When asked how he thinks about what span means, Aziz replied, "Span is just the area that it covers. It could be a plane in $\mathbb{R}^3$, it could be a line in $\mathbb{R}^3$." Most examples of the geometric category with respect to linear dependence consisted of students showing either two collinear vectors or three vectors placed head to tail to form a triangle with one vertex at the origin. For instance, Abraham constructed an example of three vectors in $\mathbb{R}^2$ and an associated sketch (see Fig. 2) to explain geometrically why the three vectors are linearly dependent.

The *vector algebraic* category captures participants' use of operations on algebraic representations of vectors to describe span and linear (in)dependence. This includes scalar multiplication, vector addition, and linear combination of vectors written as $n \times 1$ matrices, vectors designated by variables (i.e., $2\mathbf{v}+3\mathbf{w}$), and the use of the equation $A\mathbf{x}=\mathbf{b}$. Vector algebraic conceptions of span included "every vector you can make with linear combinations of the columns" (Justin) and "in order to

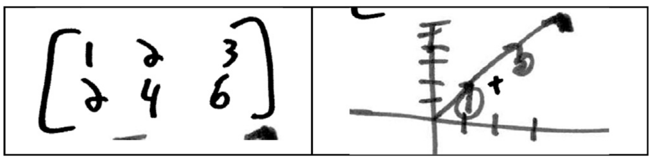

Fig. 2. Abraham's geometric explanation of three linearly dependent vectors.

span $\mathbb{R}^3$, vectors have to be different" (Giovanni). Vector algebraic conceptions of linear independence consisted of some form of the notion that only the trivial linear combination of the vectors would equal the zero vector. One participant, Abraham, described linear independence as when the equation $A\mathbf{x} = \mathbf{b}$ has one unique solution. This notion is included in this category because Abraham tended to focus on the product as a linear combination of column vectors of A rather than on the matrix as an entity that maps $\mathbf{x}$ to $\mathbf{b}$ or as a system of linear equations (Larson & Zandieh, 2013). An example of a student description of linear dependence that we coded as vector algebraic is, "But if it's linearly dependent, then there wouldn't be enough vectors because at least two of them are going to be maybe multiples of each other or just the zero vector" (Kaemon). Here, Kaemon attends to the quality of vectors being multiples of one another (as opposed to, say, vectors being collinear) to justify their linear dependence. Vector algebraic conceptions of linear dependence include when a nontrivial linear combination yields the zero vector and also the process of scaling one vector or taking a linear combination of vectors in order to produce a linearly dependent set.

Finally, the *matrix algebraic* category captures participants' explicit attention to the form or properties of a matrix, such as its size or dimension, or on procedural operations on matrices. For instance, Giovanni's statement of, "I mean I would see it as being linearly dependent because you have more columns than rows" is an example of a matrix algebraic conception of linear dependence because he attended to the form of the matrix (more columns than rows). Instances in which participants made use of matrix-oriented algorithms such as Gaussian elimination through elementary row operations were also coded as matrix algebraic conceptions. For example, Aziz stated, "If it [a 3×3 matrix] doesn't reduce to the identity then it means it doesn't span all of $\mathbb{R}^3$." In fact, in our data, this was most prevalent occurrence of a matrix algebra conception – that is, participants' reliance on the row-reduced echelon form of a matrix either equaling the identity matrix or containing a row or column of zeros. Note that in both of these quotes, Giovanni and Aziz seem to use linear dependence (for vectors in $\mathbb{R}^n$) as a property of a matrix. They may have been speaking metonymically (Lakoff & Johnson, 1980) about a single matrix as if it was itself the set of vectors in $\mathbb{R}^n$; alternatively, they may have incorrectly assigned linear dependence as a property of a matrix.

5.2. Types of mathematical activity

The construct of types of mathematical activity emerged from our grounded analysis of the data. As we analyzed students' understanding of span and linear (in)dependence in light of the concept image construct, we found ourselves continually drawn to notice the *type of activity* in which students were engaged as they responded to the interview questions. For instance, if a student spoke of span using a phrase such as "get everywhere," was that student engaged in explaining how span related to linear independence, explaining how he thought about the concept of span itself, or some other activity? As such, we identified five mathematical activities within the interview data: defining, proving, relating, example generating, and problem solving (see Table 2). We contend that considering the five mathematical activities provides insight into a student's understanding of a concept. We can consider these facets of a student's interaction with the world based on what s/he understands a concept to be. These activities do not always occur in isolation. Furthermore, an activity may arise naturally based on the interview prompt or may occur spontaneously. Here we provide descriptions and examples of each category of mathematical activity.

We use the term *defining* to mean the act of describing a concept's essential qualities. During the interviews, students were not asked to create definitions for concepts that were new to them, but rather to explain their notion of a concept that had been defined during their linear algebra course. As such, this use of defining may be of a slightly different connotation than the discipline-specific practice of defining (e.g., Zandieh & Rasmussen, 2010), but it is consistent with Zaslavsky and Shir's (2005) discussion of Tall and Vinner's (1981) notion of *personal concept definition*. For example, Giovanni, when prompted to explain in general how he thought about span, replied, "The way that I think of span is just being able to reach anywhere in, like in $\mathbb{R}^3$, like in that dimension, like you're able to get to all points in $\mathbb{R}^3$." Also, if students spontaneously (i.e., without prompting) described a concept, we coded that as a "defining" activity. An example of this is given in the next section, in which Justin spontaneously describes the concepts of linear independence and linear dependence in service of explaining how the two concepts are similar and different.

We use the term *proving* to mean the act of providing a justification to a claim. This reasoning process may be of various levels of mathematical rigor, and it may be carried out for the participant's personal conviction or to convince the interviewer. This justification may be of a variety of forms, such as a chain of reasoning or a coordination of more than one justification to support a claim. As such, we use "proving" similarly to Harel and Sowder's (1998) use of "the process of proving," which included the subprocesses of ascertaining and persuading. For example, in response to Question 1a in Fig. 1, Giovanni states, "That's false, because in order to have a 3×3 matrix that spans all of $\mathbb{R}^3$, the column vectors have to be linearly independent. And so that's how, basically that's the definition, so I think of it that way." In response to the same question, Aziz asserts, "The matrix A is a 3 by 3 . . . and since it spans all of $\mathbb{R}^3$, the columns, the column vectors of A are linearly independent." As implicitly illustrated by these quotes, the proving code is not meant to determine levels of acceptability or correctness of the student's statement. Rather, we align this code with the core of Stylianides' (2007) definition of proof in the classroom in that it is solely meant to convey when a student presents "a connected sequence of assertions for or against a mathematical claim" (p. 291).

We use the term *relating* to denote any participant activity that compares, contrasts, or explains connections between two concepts or between different interpretations of the same concept. The activities of proving and relating are similar, but distinct. We distinguish between these activities by attending to the overarching purpose of the student's utterance. For instance, a participant might provide a statement of two concepts' relationship in support of claims during a proving activity or engage in proving activity to support why he or she feel a specific relationship exists. For the purposes of our analysis, we restrict the code of *relating* activity to the participant's statement of a general relationship. The activity of *example generating* denotes when participants create cases of certain concepts or properties (e.g., a set of three linearly dependent vectors in $\mathbb{R}^3$), which is consistent with Zazkis and Leikin's (2008) use of the term "example." As with the other activities, this may be prompted by the interviewer explicitly or spontaneously done by the interviewee.

Finally, the activity of *problem solving* is engaging in calculation or reasoning with a specific goal to determine a previously unknown result. In this paper, "problem" can be considered according to either of Schoenfeld's (1992) definitions – as a routine exercise that requires completion or as a perplexing and difficult question – the latter of which is more consistent with the NCTM Process Standard definition of problem solving (NCTM, 2000). This category is used to describe activity in which students carry out a sequence of mathematical operations to find a result that they might then interpret as providing information about a given situation. For example, to glean more information about a student's conception of span or linear (in)dependence, the interviewer occasionally would pose a closed-ended problem about the span or linear (in)dependence of a specific set of vectors to a student and ask him to solve it. The activity of problem solving did not occur as frequently as the other codes within the data set; we attribute this to the nature of the interview questions, which were created to have students engage in justifying a true/false conclusion and explaining their understanding of the related concepts. Thus, by virtue of the interview questions, the mathematical activities of defining, proving, and relating were most common in the data set.

5.3. *Coordinated analysis: using the two types of categorization*

We first illustrate, with one short section of transcript, how we made use of the two categorizations within our analysis. The given transcript comes from a portion of Justin's interview in which he was describing the differences and similarities between linear independence and linear dependence:

Justin: They're exactly the same as in it's where you can get, it's just different
rules in how you get there, and if you can get back and whatnot.
Interviewer: Can you say some more about the difference in the rules?
Justin: The difference is huge! You know, uh, the difference is that with
independence, you can only go farther away and maybe kind of
sideways, but you can't come back. Dependence, you can make it back
to where you started.
And, um, so even though I know they're not the same at all really, it's
just they are the same because it's describing where you can get to, just
it says different things about them.

In lines 1 and 2, we code Justin's discourse as the mathematical activity of *relating*, and the underlined portions indicate an image category of *travel.* In response to the interviewer's inquiry in line 3, Justin elaborates. In line 4, Justin *relates* linear independence and dependence by saying "the difference is huge." To substantiate that claim, he then engages in the activity of *defining* (lines 5–7). That is, he gives what, to him, is an essential quality of linear independence by stating, "You can only go farther away and maybe kind of sideways, but you can't come back." He then engages similarly for linear dependence, stating, "You can make it back to where you started." Both of these activities of defining are marked with language consistent with *travel* imagery, as indicated by the underlined portions of lines 5–7. We note here that Justin's descriptions of linear independence and linear dependence align with the formal definition of the concepts, specifically that "mak[ing] it back to where you started" is analogous to finding a nontrivial solution to the homogeneous equation and that "go[ing] farther away" and not "com[ing] back" corresponds to having only the trivial solution to the homogeneous equation. Finally, we code Justin's discourse in lines 8–10 as *relating* as he concludes his explanation of the similarities and differences between linear independence and dependence, all which again using language consistent with the *travel* image category.

The remainder of Section 5 is dedicated to illustrating how coordinated analyses of students' concept images with their activity provides a compelling framework for gaining nuanced insight into students' understanding of mathematical concepts. To illustrate this, we consider how three participants (Abraham, Kaemon, and Aziz) engaged in Question 1a (see Fig. 1). First, we show how Abraham's problem solving activity provides insight into how he related his varied conceptions of span. We then show how Kaemon's restricted conceptions of span and linear dependence possibly prevented him from meaningfully relating the two concepts. Finally, we describe Aziz's process of "refiguring out" how his notions of span and linear dependence are related through an example generating activity.

5.3.1. Abraham

We begin by providing the first portion of transcript of Abraham's initial response to Question 1a:

Abraham: That would be false. Let's see. [7-sec pause] So if, the way I think of it is, if it, if it's spanning a 3 by 3 matrix [draws the outer brackets for a matrix], um, then it's going to have like, you know, three pivot positions here [writes three ones in on the diagonal of his matrix]. And for like a square matrix, I just think like if this is three pivots in each row, then it's also going to be, automatically going to be 3 pivots in each column. And that way you're always going to have a linearly independent set of. . .of, um, like *x* equals something, *y* equals something, *z* equals something, because of that. And then that's, so that's going to be, basically a unique solution for every output. So let's see, so [writes $A\mathbf{x} = \mathbf{b}$], for every $\mathbf{b}$ there's a unique $\mathbf{x}$ vector.

After (correctly) commenting that the given statement is false, Abraham began by writing an empty matrix with 1's on the main diagonal (lines 1–4) and claimed that "if it's spanning a 3×3 matrix" (line 2), there would be three pivots down the diagonal. We note that Abraham did not directly address whether the matrix is given in this form or whether it must be manipulated to fit this form (via elementary row operations). He went on to claim that such a set of vectors would be linearly independent because the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution (lines 6–11). We categorize these initial statements as the mathematical activity of proving. We further coded this as matrix algebraic because his language indicates that Abraham used matrix algebraic conceptions of span and linear independence within his justification. Later, however, while explaining why he drew the three 1's down the diagonal of the matrix, Abraham said, "if it's spanning something, it kind of needs to, I think of it like it needs to go in every direction." Here, Abraham provided an essential quality of span, so this was coded as a defining activity. In addition, the language of this defining activity indicates a travel conception of span, distinct from his previous use of a matrix algebraic conception of span.

The interviewer probed Abraham to further explain the connection between the 1's and "going in every direction":

Interviewer: How is it that those 3 pivot positions allow you to do those 3 directions?

Abraham: I go in this direction and then I can kind of pivot up, you know to go to this direction . . . so I can kind of go any direction, based on these 3 pivots, like pivoting in each direction, to get to any point.

Interviewer: That's interesting, I like that language. So like if, say you had an example like, I don't know, 3,4,5, can you explain how you would use that language of pivoting to get to 3,4,5?

Abraham: I'm, I'm saying that I could use, you know, three [writes "3<1,0,0>"] of that one, a linear combination of this guy [writes "+4<0,1,0>"] and then this [writes "+5<0,0,1>="]. I could write that out as a linear combination of 3,4,5 [writes "<3,4,5> to the right of the equals sign]. And so you can see, I can do that for a lot of different vectors, and so if I think of it in these terms, the basis vectors, that I could, you know, whatever I put here [points to the '3' in his first vector's scalar], is going to be at the top [points to the '3' in the <3,4,5> vector]. So I can get to any point there, I can get to any point there [repeats the analogous gestures for the "4" and '5'] and so forth to get to that point.

Within lines 19–28, Abraham showed, for the given vector $\begin{bmatrix}3\\4\\5\end{bmatrix}$, how the columns in the matrix with spanning column vectors (standard basis vectors) could be used in linear combination "to get to that point." In lines 23–28, Abraham alluded to being able to do this for any given vector. The mathematical activity within lines 23–28 is coded as problem solving because the vector $\begin{bmatrix}3\\4\\5\end{bmatrix}$ is a specific instantiation of a general vector in $\mathbb{R}^3$, and because Abraham engages in his demonstration through an algorithmic process. What is notable about this process is that Abraham's actions focused on each column of the matrix as an individual vector, whereby he calculated a scalar multiple for each vector so that the resulting linear combination would yield the vector $\begin{bmatrix}3\\4\\5\end{bmatrix}$. That is, Abraham's focus shifted from matrix algebraic and travel conceptions of span (in lines 1–11) to a vector algebraic conception of span, within which his actions were oriented around linear combinations of vectors (lines 19–28).

Through the problem solving activity, Abraham demonstrated relationships between his travel and matrix algebraic conceptions of span. In Abraham's case–although the immediate purpose of his work was to solve a closed-ended problem–he used a linear combination of vectors to relate how the matrix with spanning column vectors (his matrix algebraic conception of span) indeed reaches everywhere in $\mathbb{R}^3$ (his travel conception of span). We view this as Abraham's active coordination of

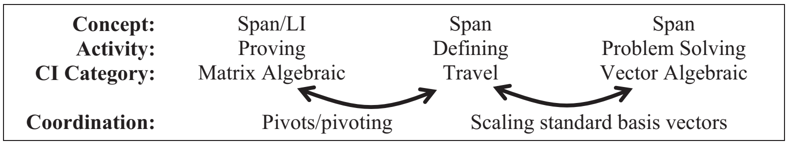

Fig. 3. Diagramming Abraham's coordination of span and linear independence.

his many conceptions of span, in which the various parts of his understanding come together to produce a meaningful (to Abraham) way for him to think about what it means for vectors to span a vector space.

This analysis is supported by the alignment of Abraham's different conceptions across activities, which is illustrated in Fig. 3. In Fig. 3 we organize Abraham's response, showing how his proving activity (excerpt 1) is coordinated with his defining activity (quoted above) through the notion of pivots. His defining activity is also coordinated with his problem solving activity (excerpt 2) in which he scaled the standard basis vectors to demonstrate how he coordinated his travel conception of span with his Vector Algebraic conception. Thus, by using a framework of dual analysis to parse apart Abraham's engagement in this specific mathematical question, we gain deeper insight into his conceptions of span and linear independence. Through notating the ways in which he fluidly moved between proving, defining, and problem solving via coordinated vector algebraic, matrix algebraic, and travel conceptions of span and linear independence, the characterization, nuance, and complexity of his rich understanding comes to the fore. We note this is in line with Stewart and Thomas's (2010) recommendation that a balanced reliance on embodied notions (such as travel concepts) can help facilitate the development of concepts at a more formal level.

5.3.2. *Kaemon*

In his initial response to Question 1a, Kaemon said the following:

Kaemon: Ok. [6-sec pause] Uh, I say that this is false because the key is that it's 3 by 3 and it's, it says you know that the columns of *A* span $\mathbb{R}^3$, so, um, like the minimal amount of vectors you could have to span $\mathbb{R}^3$ is at least three. But if it's linearly dependent, then there wouldn't be enough vectors, because at least two of them are going to be maybe multiples of each other or just the zero vector. So that's why I say it is false.

Kaemon engaged in this proving activity using a matrix algebraic conception of span (focusing on the size of the matrix, lines 1–3) and a vector algebraic conception of linear dependence (how individual vectors relate to each other, lines 4–6). The interviewer then asked Kaemon to generate an example of a matrix with linearly dependent column vectors, to which he responded with the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. This example generating activity shows Kaemon attended to the relationships between the individual vectors, also indicating a vector algebraic conception of linear dependence. When asked to explain why these column vectors are linearly dependent, Kaemon stated that these vectors are "all on the same line" and added, "they won't be able to span, it's dependent." This first statement shows that Kaemon is attending to a geometric conception of linear dependence when discussing the example he generated. The second statement is coded as relating activity, although there is no clear indication of what conception of span and linear dependence Kaemon is attending to when stating this relationship.

The interviewer then asked Kaemon to define span, prompting the following dialogue:

Kaemon: I think span is, I think it's all the possible linear combinations of the matrix. So since it's, like you're on this one line, you can't really get all the combinations that are in these quadrants [moves pen around the coordinate system], so that's how I think of it.

Interviewer: Ok. So you just said a second ago the way you think of span is all of the linear combinations of the columns? That's good. Um, how do you think about linear dependence in general?

Kaemon: Um, dependence for me is just, first I just try to look with, if there's a matrix to see if it's, like, if it's already reduced, then I see if there's a variable, or I see that they're multiples, that they're 0 vector, just something to show it's dependent. And then if I can't find that, like, right away, then maybe I'll try to then, I don't know, try to reduce it, whatever, just until I could figure it out.

We categorize lines 7–10 as defining activity indicative of a vector algebraic conception of span. This is a slight shift from Kaemon's previous focus on the dimensions of the matrix (i.e., the number of rows and columns) in lines 1 and 2. This dialogue also indicates a shift from vector algebraic and geometric conceptions of linear dependence to a matrix algebraic conception, because his focus changed from scalar multiples of vectors to the row-reduced echelon form of a matrix (lines 14–19). Notice,

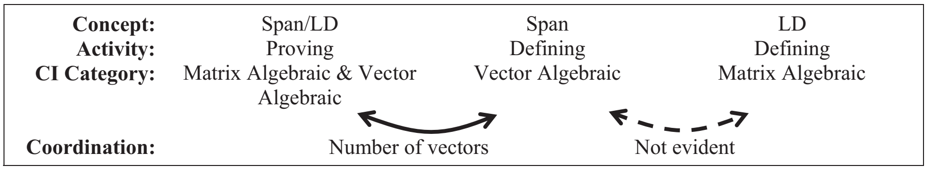

Fig. 4. Kaemon's uncoordinated understanding of span and linear dependence.

though, that although Kaemon's defining activity explicitly addresses if the column vectors are scalar multiples or include a zero vector (line 16), he requires the elementary row operations involved in row reduction before considering vector algebraic relationships between the column vectors. Kaemon similarly defined linear independence in the context of a row-reduced matrix. We note that this definition of linear dependence (and hence linear independence) is restricted to a single algorithm carried out on a specific, given matrix.

To summarize Kaemon's response in total, he correctly answered that the statement in Question 1a was false, citing the number of vectors to support his response, although he never explicitly stated a broad relationship between the concepts of span and linear independence (lines 1–6). Kaemon continued to demonstrate multiple conceptions of span and linear (in)dependence but failed (at least during the interview) to meaningfully coordinate these conceptions through any of his mathematical activity. We attribute part of this limited ability to coordinate the two concepts to Kaemon's matrix algebraic conceptions of linear independence and linear dependence. Specifically, Kaemon's definition of each concept necessitated a given matrix upon which he could carry out elementary row operations. We note here that Kaemon alluded to, but never engaged in, problem solving activity, which may have provided deeper insight into how he might have coordinated his conceptions (as in Abraham's case). Also, consider Kaemon's three conceptions of linear dependence: collinearity when discussed geometrically, scalar dependence when discussed vector algebraically, and inclusion of the zero vector in a row-reduced matrix when discussed matrix algebraically. These are restricted forms of the mathematical definition, for instance, neglecting coplanarity and its higher dimensional analogs or restricting the context to a problem-solving situation, dependent on the row reduction algorithm.

We use Fig. 4 to highlight how Kaemon's conceptual understanding is not fully coordinated through his various activities. For instance, his proving activity (excerpt 1) is coordinated with his defining activity (quoted above) through the notion that there are not enough vectors to span the space if they are linearly dependent. This might initially be viewed as a fruitful coordination. However, by attending to his defining activity (excerpt 2), we see that his reasoning is restricted to this seemingly rote connection, exemplified by the lack of any meaningful coordination between his two definitions. Specifically, Kaemon is unable to articulate or demonstrate any coordination between his vector algebraic definition of span and his matrix algebraic definition of linear dependence, which we denote with a dashed arrow.

5.3.3. *Aziz*

In our discussion of Aziz, rather than concentrating on his response to Question 1a, we focus on a specific episode wherein he established a relationship between linear dependence and span. When asked how he thought of the idea of span in general, Aziz responded, "Span is just the area that it covers, it could be a plane in $\mathbb{R}^3$, it could be a line in $\mathbb{R}^3$." The interviewer asked him to "give an example of one where you said it's like a plane in $\mathbb{R}^3$," Aziz described a specific type of linear dependence in which "two, three completely different matrices (sic) that are not in combinations of each other, but they combine in a way to equal zero." He was contrasting this type of linear dependence with an earlier example he had generated that contained three vectors, two of which were collinear. Formally speaking, Aziz was trying to generate a set of three coplanar vectors in $\mathbb{R}^3$, none of which is pair-wise collinear with another. Aziz then engaged in the following example generating activity:

Aziz:	So *C* equals say 2,2,2; this [a second vector] is 1,2,3. And then, you know, I don't know how to calculate it right now.
Interviewer:	Ok. What would you, just off the top of your head, what would you do—
Aziz:	2, 2**v**, $2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.
Interviewer:	Hmm. Ok I—
Aziz:	So that way you can move one away on one vector, second way and then take the third one back to the origin.

Initially, Aziz attempted to generate specific column vectors in $\mathbb{R}^3$ that "are not in combinations of each other" but stalled after his first two vectors (lines 1 and 2). He then attempted to describe what such a linear combination would be, but he seemed to be unable to use this algebraic representation with the first two vectors he had produced to generate the third vector (line 4). Throughout this example generating activity, Aziz's discussion focused on linear combinations of vectors, first representing them as columns in a 3×3 matrix, and then writing them as the abstract vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. This indicates a vector algebraic conception of linear dependence under two different representations. Aziz then used the language of "move one away" and "take the third one back" to describe the linear combination (lines 6 and 7), indicating a travel conception of linear dependence. As Aziz used the travel imagery, he traced his finger over a drawn set of axes in a triangular pattern, which he later drew (Fig. 5).

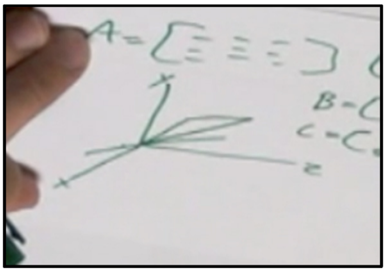

Fig. 5. Aziz's geometric representation of three linearly dependent vectors.

When asked whether the vectors he described span $\mathbb{R}^3$, Aziz was perturbed (lines 8–10). Aziz's mathematical activity shifts here from example generating to relating activity (lines 14 and 15) and proving activity (13 and 14). He outlined a seeming contradiction, which we reorder and paraphrase for clarification: vectors that span $\mathbb{R}^3$ "move in three different directions" (9 and 10), a set of linearly dependent vectors does not span $\mathbb{R}^3$ (14 and 15), and this example shows vectors that "move in three different directions and get back to the origin [are linearly dependent]" (12–14). Notice that the second of these statements is possibly false if the set under consideration contains more vectors than the dimension of the vector space.

8	*Aziz*:	They're linearly dependent. Um. . .that's a problem I always thought,
9		because if it's they move in 3 different directions, they should technically
10		span $\mathbb{R}^3$, but I never got clarification on that.
11	*Interviewer*:	Say a little bit more about what's confusing to you about that?
12	*Aziz*:	Because if they uh, they move different directions from each other,
13		[Interviewer: Hm-mm] but they're linearly dependent, because you can
14		use a combination of all 3 to get back to the origin. So linear dependence
15		means it doesn't span all of $\mathbb{R}^n$. Right?

We see that Aziz was perturbed by this question because he questions his own statement (line 15). Indeed, when pressed to explain why he thought this, he responded, "That's just what I think, I don't know why, I don't know if it's even true." Aziz went on to make sense of this seeming contradiction by drawing the vectors and pointing out that the linearly dependent vectors he generated move three different directions on the same plane and not outside of that plane.

16	Aziz:	So you can take this one [traces over the first vector (Fig. 3)] and that
17		one [traces over the second vector], like end up here [draws an extension
18		off his first vector] somewhere. And then take this one [points to the third
19		original one he drew, then draws in a line from the end of his two vectors
20		back to the origin], vector 3, to end up back there. But it technically spans,
21		no, makes it a plane in $\mathbb{R}^2$, $\mathbb{R}^3$. I got it, I figured it out.
22	Interviewer:	Can you explain to me what you just figured out?
23	Aziz:	Because they move on the same plane, if it's linearly dependent, linearly
24		independent in $\mathbb{R}^3$ it has to move three different ways. But if it's linearly
25		dependent but they're a combination, it must be that they move on the same
26		plane in 3 different directions, but not out of that plane. So that's why
27		you're able to get back to the origin.

Drawing a distinction between two types of "mov[ing] three different ways" (line 24), Aziz was able to reason that his generated vectors did not span $\mathbb{R}^3$. This understanding came to the fore during Aziz's example generation using geometric representations combined with travel language. Specifically, Aziz described spanning $\mathbb{R}^3$ as moving in three different directions, and he described linear dependence as being able to "get back" to the origin. Generating this set of vectors allowed Aziz to shift toward proving (lines 23–27) the relationship that he had earlier questioned (lines 8–10) by coordinating his geometric/travel conception of span and his geometric/travel conception of linear dependence. This is a coordination that eluded him earlier, as he used vector algebraic representations for vectors satisfying the same type of coplanar linear dependence.

In this episode, Aziz began with an example generating activity that used a vector algebraic conception of span (Fig. 6). After shifting his example generating activity to draw on a travel conception of linear dependence, Aziz was perturbed by a seeming contradiction to a relationship he held between linear dependence and span. Finally, Aziz made sense of this contradiction through his proving activity that drew upon a travel conception and geometric conception of each. Throughout the episode, Aziz's activity became increasingly productive. He began to draw on a different concept image as he shifted from his stalled example generating activity to relating activity. This, along with his attention to the geometric aspects of his diagram, supported a shift into successful (from his perspective) proving activity. Distinguishing between Aziz's activities allows us to coordinate his various evoked concept images and how they lent themselves toward this more productive way of thinking about linear dependence and span, which allowed him to resolve his perturbation. We see that this activity was

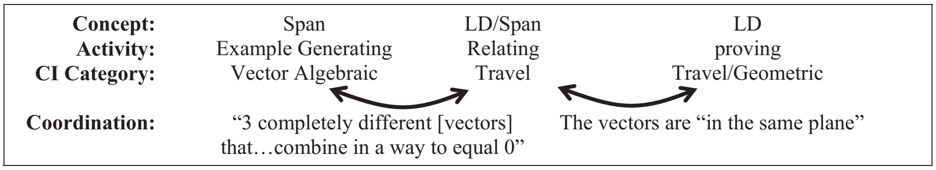

Fig. 6. Aziz's coordination of span and linear dependence.

meaningful to Aziz when he stated, "Yeah, I learned something just by figuring out something . . . I figured out that, I kind of knew it before, but I kind of lost sight of why it was, but I refigured that out again, that the combination is . . . they move on the same plane, which makes sense."

6. Discussion

This paper describes our work in classifying students' conceptions of span and linear (in)dependence and investigating how students use these conceptions to reason about relationships between span and linear (in)dependence. This research endeavor led us to developing the dual categorization framework in which we utilize both characterizations of students' concept images of span and linear (in)dependence and students' mathematical activity to provide insight into how these conceptions might be coordinated for students. We note that the concept image categories that arose may be an artifact of the type of instruction and curriculum that these students experienced. As such, we would expect that additional or alternative categories for student conceptions would arise from the data with different data sources (such as whole class discussion rather than individual interview data), different participants (such as students from a more advanced, proof-based linear algebra course), or different content areas (such as abstract algebra). We also note that the five types of mathematical activity within our framework – defining, proving, relating, example generating, and problem solving – are not meant to be exhaustive; rather, categorization of student activity depends on data and will likely be extended, or altogether different, in future investigations. We view the current work as generalizable through its potential to facilitate researchers' attention to both activity and conceptual understanding in different domains so that it may provide similar insight into students' engagement in and coordination between other contexts and content, respectively.

We conclude with a summary of the results in light of the dual categorization framework, directions for future research, and a discussion of instructional implications.

6.1. Framework of dual categorization

We have found the coordination of these two facets of student responses that is afforded us through the dual categorization framework to provide deeper, richer descriptions of student's understanding of linear (in)dependence and span than would have been possible by attending only to concept imagery or activity. Specifically, our results using this framework have shown how students' engagement in different activities affords useful insight of the varied facets of the students' conceptions. The dual categorization allowed us to organize our participants' responses so that we were able to draw out more subtlety in how their various conceptions might be coordinated. In that sense, the categorization is extended beyond taxonomy, toward a framework that lends itself toward deeper insight of student understanding. Consequently, we feel that this work extends the existing literature to incorporate not only how students might conceive of linear (in)dependence or span, but also how these different conceptions might be coordinated by the student in activity.

Abraham's defining and proving activity, while mathematically appropriate and sound, did not independently provide insight into how his varied conceptions of span were coordinated. It was not until his problem solving activity that we were able to characterize his various notions of span – more precisely, until he was able to coordinate his matrix algebraic and travel conceptions of span through problem solving activity. Similarly, we were able to better characterize Aziz's conceptual understanding of linear dependence and span by attending to and parsing out his shifts between activities. For instance, Aziz's proving activity not only foregrounded his geometric conception of co-planarity that was not readily evident during his defining and example generating activity but also provided an opportunity for coordination of his travel conceptions of linear dependence and span. In these cases, the dual framework allowed us to contextualize different instances of participants' evoked concept images, affording an overarching coordination of their conceptual understanding and activity. Conversely, Kaemon's defining activity demonstrated a restricted understanding of linear independence and linear dependence – restricted in the sense that his understanding of the concepts depended on a specific context in which he could perform an algorithm to test for a specific trait in the row-reduced matrix. This is supported by the fact that his defining activities did not allow for any further coordination of his notions of the two concepts beyond his assertion using the number of vectors in his proving activity. From the observed activity during the interview, we are unable to determine if Kaemon's conceptions of linear independence and linear dependence are informed by more meaningful connections (as was seen with Abraham); however, the context-dependent nature of Kaemon's defining activities suggests that he was unable to coordinate such connections across the concepts in different contexts.

Furthermore, each of our participants used various conceptions when engaging in different activities. This is consistent with the Tall and Vinner's (1981) discussion of evoked concept image and is distinct from the notion of using different representations (e.g., vectors as arrows, n-tuples, or abstract $\mathbf{v}$) in different activities. In other words, the concept images drawn on by the participants afforded specific ways of reasoning and communicating that other concept imagery did not provide during the interview, rather than merely lending alternative representations or inscriptions. For instance, Aziz used n-tuples and abstract representations with his vector algebraic conception of linear dependence, but he used a geometric representation with his travel conception of linear dependence–all of which occurred during example generating activity. It was not until Aziz represented linear dependence geometrically and drew on his travel conceptions of span and linear dependence that he was able to realize and eventually make sense of a seeming contradiction to his previous understanding of the relationship between the two concepts. Considering this type of coordination alongside analyses of Abraham and Kaemon suggests that deeper, more productive understanding is contingent on not only drawing on multiple concept images, but coordinating these images within and across one's various purposeful activities.

6.2. Future research and instructional implications

As previously mentioned, the five types of mathematical activity within our framework were determined from analysis of our data set. Analyses of other types of data – for instance, from the classroom – would likely give rise to additional types of mathematical activity. As such, our future work involves a further examination and refinement of the mathematical activity categories used to gain insight into students' conceptions. We also plan to examine additional data (e.g., classroom video data at the whole-class and small-group level, mid-semester interviews) of these same five students to gain a more complete analysis of their understanding. Given the power that a coordinated analysis has provided within this data set, we have begun research that further explores this dual usage of concept imagery and activity. In particular, we designed a new interview protocol for semi-structured individual interviews that is composed of tasks that (a) purposefully aim to engage students in the five various mathematical activities, and (b) make use of a variety of vector representations (e.g., arrows, specific vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, or abstract vectors such as $\mathbf{v}$ and $\mathbf{w}$). The mathematical content of this research focuses on students understanding of linear independence in various vector spaces such as $\mathbb{R}^n$ and function spaces, as well as student understanding of relationships between span and linear independence (Plaxco, Wawro, & Zietsman, 2014). As such, Question 1a (analyzed in this paper) is included in the new interview protocol. We conducted interviews with seven, first-year, honors STEM majors, and the analysis of these data is ongoing.

We view the current work as a first step toward a better understanding of students' development of these concepts and pedagogical strategies to support such development. This work provides examples of both productive and unproductive student thinking. For instance, Abraham and Aziz were able to reason about concepts in different activities by drawing on varying concept imagery and also leveraging specific aspects of their ways of thinking in more coordinated ways. Conversely, Kaemon was able to describe span and linear independence using different concept images and in different activity, but was unable to communicate about how these different conceptions informed each other across activity. This informs teaching practices by providing insight into how students might make sense of concepts by engaging in different mathematical activity. We suggest that instruction and assessment should not only draw on different concept images, but also incorporate such imagery and associated reasoning in a variety of activities that provide opportunities for students to develop a richer, better coordinated conceptual understanding. This should also allow instructors to assess broader connections that students might have developed. It is our hope that future research using the dual categorization framework will provide further insight into the teaching and learning of these concepts.

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References

Aydin, S. (2014). Using example generation to explore students' understanding of the concepts of linear dependence/independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *45*(6), 813–826.

Bernard, R. H. (1988). *Research methods in cultural anthropology*. Newbury Park, CA: Sage Publications.

Blumer, H. (1969). *Symbolic interactionism: Perspectives and method*. Englewood Cliffs, NJ: Prentice-Hall.

Bogomolny, M. (2007). Raising students' understanding: Linear algebra. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st conference of the international group for the psychology of mathematics education, Vol. 2* PME, Seoul, (pp. 65–72).

Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307–333). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*, 175–190.

Denzin, N. K. (1978). *The research act: A theoretical introduction to research methods*. New York: Mcgraw-Hill.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction*, *10*(2–3), 105–225.
Dogan-Dunlap, H. (2010). Linear algebra students' modes of reasoning: Geometric representations. *Linear Algebra and its Applications*, *432*, 2141–2159.
Dorier, J.-L. (Ed.). (2000). *On the teaching of linear algebra*. Dordrecht: Kluwer Academic Publisher.
Dorier, J.-L., Robert, A., Robinet, J., & Rogalski, M. (2000). The obstacle of formalism in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 85–124). Dordrecht: Kluwer Academic Publisher.
Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275–282). Dordrecht, the Netherlands: Kluwer Academic Publishers.
Ertekin, E., Solak, s., & Yazici, E. (2010). The effects of formalism on teacher trainees' algebraic and geometric interpretation of the notions of linear dependency/independency. *International Journal of Mathematical Education in Science and Technology*, *41*(8), 1015–1035.
Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer Academic Publishers.
Glaser, B., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago, IL: Aldine Publishing Company.
Harel, G. (1989). Learning and teaching linear algebra: Difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics*, *11*(1), 139–148.
Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, III* (pp. 234–283). Providence, RI: American Mathematical Society.
Hillel, J. (2000). Modes of description and the problem of representation in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 191–207). Dordrecht, The Netherlands: Kluwer Academic Publishers.
Kú, D., Oktaç, A., & Trigueros, M. (2011). Spanning set and span: An analysis of the mental constructions of undergraduate students. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 14th annual conference on research in undergraduate mathematics education, Vol. 1* Portland, OR, (pp. 176–186).
Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago: The University of Chicago Press.
Larson, C. (2010). *Conceptualizing matrix multiplication: A framework for student thinking, an historical analysis, and a modeling perspective*. Dissertation Abstracts International, 71-09. Retrieved 23.05.12 from Dissertations & Theses: Full Text (Publication No. AAT 3413653).
Larson, C., & Zandieh, M. (2013). Three interpretations of the matrix equation $Ax = b$. *For the Learning of Mathematics*, *33*(2), 11–17.
Lay, D. C. (2003). *Linear algebra and its applications* (3rd ed.). Reading, MA: Addison-Wesley.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
Plaxco, D., Wawro, M., & Zietsman, L. (2014). Student understanding of linear independence of functions. In T. Fukawa-Connelly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th annual conference on research in undergraduate mathematics education* Denver, CO, (pp. 992–997).
Rasmussen, C., & Kwon, O. N. (2007). An inquiry-oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior*, *26*, 189–194.
Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259–281.
Rogalski, M. (2000). The teaching experimented in Lille. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 133–149). Dordrecht: Kluwer Academic Publisher.
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In *Handbook of research on mathematics teaching and learning*.
Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). Dordrecht, The Netherlands: Kluwer Academic Publishers.
Stewart, S., & Thomas, M. O. J. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 173–188.
Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.
Tall, D. O. (2004). The three worlds of mathematics. *For the Learning of Mathematics*, *23*(3), 29–33.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.
Trigueros, M., & Possani, E. (2013). Using an economics model for teaching linear algebra. *Linear Algebra and Its Applications*, *438*(4), 1779–1792.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–79). Dordrecht, the Netherlands: Kluwer Academic Publishers.
von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Bristol, PA: Falmer Press.
Wawro, M. J. (2011). *Individual and collective analyses of the genesis of student reasoning regarding the Invertible Matrix Theorem in linear algebra*. (Doctoral dissertation). Available from ProQuest Dissertations and Theses database (Order No. 3466728).
Wawro, M., Sweeney, G., & Rabin, J. M. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics*, *78*, 1–19.
Wawro, M., Larson, C., Zandieh, M., & Rasmussen, C. (2012). A hypothetical collective progression for conceptualizing matrices as linear transformations. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th annual conference on research in undergraduate mathematics education* Portland, OR, (pp. 465–479).
Wawro, M., Rasmussen, C., Zandieh, M., Sweeney, G., & Larson, M. (2012). An inquiry-oriented approach to span and linear independence: The case of the Magic Carpet Ride sequence. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *22*(8), 577–599.
Zandieh, M., & Rasmussen, C. (2010). Defining as a mathematical activity: A framework for characterizing progress from informal to more formal ways of reasoning. *Journal of Mathematical Behavior*, *29*, 57–75.
Zandieh, M., Ellis, J., & Rasmussen, C. (2012). Student concept images of function and linear transformation. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th annual conference on research in undergraduate mathematics education* Portland, OR, (pp. 320–328).
Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education*, *36*(4), 317–346.
Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: The case of a square. *Educational Studies in Mathematics*, *69*, 131–148.