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Inverse, composition, and identity: The case of function and linear transformation

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ABSTRACT

In this report we examine linear algebra students' reasoning about composing a function or linear transformation with its inverse. In the course of analyzing data from semi-structured clinical interviews with 10 undergraduate students in a linear algebra class, we were surprised to find that all the students said the result of composition of a function and its inverse should be 1. We examined how students attempted to reconcile their initial incorrect predictions, and found that students who succeeded in this reconciliation used what we refer to as "do-nothing function" and "net do-nothing function" reasoning. We provide examples of these patterns of reasoning, and propose explanations for why this reasoning was helpful. We also discuss possible sources for this incorrect prediction, and provide implications for classroom practice.

1. Introduction

The concept of function is central to much of secondary and undergraduate mathematics, and there is a robust body of literature examining the nature of students' conception of function (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Dubinsky & McDonald, 2001; Harel & Dubinsky, 1992; Monk, 1992; Sfard, 1991, 1992). One important context where functions appear is in linear algebra: linear transformations are functions from one vector space to another, often $\mathbf{R}^n$ to $\mathbf{R}^m$, with particular linearity properties (they preserve addition and scalar multiplication). However, the literature on student understanding of linear transformations is relatively sparse, and focuses largely on student difficulties with linear algebra without directly examining function conceptions per se (e.g., Dorier, Robert, Robinet, & Rogalski, 2000; Dreyfus, Hillel, & Sierpinska, 1998; Hillel, 2000; Portnoy, Grundmeier, & Graham, 2006; Sierpinska, 2000).

Moreover, little attention has been paid in the literature to the extent to which students make connections between function in algebraic contexts and transformation in the context of linear algebra (though for recent work in this area, see Zandieh, Ellis, & Rasmussen, 2013). This is of particular interest because prospective high school teachers typically take linear algebra, but how their studies affect their understanding of function is unclear. For example, perhaps their study of transformation in linear algebra actually has a negative effect on their understanding of function. On the other hand, perhaps their study of transformation in linear algebra reinforces and enriches their understanding of function. In either case, their study of linear algebra carries important consequences for their future teaching of secondary school students.

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In order to explore this relationship, we conducted interviews with undergraduate linear algebra students on the extent to which they do or do not construe similarity between function and transformation. These interviews covered a wide range of specific ideas, such as one-to-one, onto, composition, and invertibility. When we conducted and analyzed these interviews, we were surprised to find that all the students made an incorrect prediction about the result of composition of a function with its inverse. In this paper we unpack this surprising result and examine the reasoning of the students who were and were not able to reconcile their incorrect prediction.

2. Theoretical background

Many researchers (e.g., Dubinsky, 1991; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard, 1991, 1992; Zandieh, 2000) have discussed the dual nature of the function concept. Sfard (1991), for example, asserts that many abstract mathematical concepts, function among them, can be understood either *operationally*, as processes, or *structurally*, as objects. The operational conception is couched in the language of "processes, algorithms and actions," whereas the structural conception speaks of "static, integrative" objects (p. 4). These two distinct yet complementary aspects of a concept are related reflexively: every process needs objects to operate upon, and processes can come to be understood as objects that can then be acted upon by other processes.

In the framework of Sfard (1991), the development of a concept typically proceeds from operational to structural, passing through three stages called *interiorization*, *condensation*, and *reification*. First, during the *interiorization* stage, the student explicitly performs a process on objects that are already familiar; for instance, students learning about functions may compute tables of functional values by explicitly evaluating functional expressions at particular numbers. Next, in the phase of *condensation*, the student gradually increases in the ability to reason about the process as a coherent whole. In a sense, the procedure becomes a "black box" that objects can be pushed through without attention to the internal workings. Finally, and usually quite suddenly, the concept undergoes a *reification* and becomes an object in its own right, able to be operated upon by other processes.

Another account of the development of the function concept is given by action, process, object, and schema (APOS) theory (Asiala et al., 1996; Breidenbach et al., 1992; Dubinsky & Harel, 1992; for a more thorough treatment of the relationship between APOS theory and Sfard's account, we refer the reader to Zandieh, 2000). In this framework, the action and process stages are similar to Sfard's description of operational conceptions of function: students with an *action* view of function operate with functions by simply carrying out calculations on specific numbers, or interpreting the graph of a function as simply a curve or a fixed object in the plane; an underlying interpretation of function as a relationship between two sets is absent. Students exhibiting a *process* view of function are able to think of a function as receiving inputs, performing operations thereon, and returning outputs. With a process conception of function, students can chain two processes together to reason about their composition, or reverse a process to reason about its inverse.

Common student difficulties with composition and inverting are often linked to students' inability to go beyond an action conception of function (Dubinsky & Harel, 1992). Even (1990, 1993) found that prospective high school teachers without a modern view of function often do not view the result of composition of functions as a function itself; they thus lack understanding of the particular strength of the function concept. While several studies have examined students' understanding of composition and inverse functions, including the development of items to assess students' ability to compose a function with the inverse of another function (Carlson, Oehrtman, & Engelke, 2010), little attention has been paid in the literature to the specific case of composition of a function with its inverse.

The development of the function concept from process to object is not without its difficulties. Sfard (1992) notes that many students develop the "semantically debased conception" she refers to as *pseudostructural* (p. 75). Students exhibiting a pseudostructural conception may, for instance, regard an algebraic formula as a thing in itself divorced from any underlying meaning, or a graph as detached from its algebraic representation or the function it represents. Zandieh (2000) describes a pseudostructural conception as a gestalt: that is, "a whole without parts, a single entity without any underlying structure" (p. 108). In the language of Dubinsky, a pseudostructural conception of function is an object view that cannot be "de-encapsulated," or unpacked to get at the underlying process from which it arose.

3. Methods and background

The subjects of this study were undergraduate students from a linear algebra course at a large public university in the southwestern United States. This course is a sophomore-level course with Calculus I as a prerequisite. It is taken by students in a wide variety of majors, including mathematics, economics, engineering, and computer science. It covered a fairly standard set of topics, including solving systems of linear equations, linear transformations, vector spaces and more abstract linear spaces, determinants, orthogonality, and basic eigentheory. This course was taught by an instructor who was very familiar with the literature on student thinking in linear algebra. The instructor was a member of the research team, but is not one of the authors of this paper.

Several days after the class's final exam, 10 student volunteers participated in semi-structured hour-long clinical interviews (Ginsburg, 1997) examining their reasoning about the similarities and differences between function and linear transformation. These volunteers were reasonably representative of the students in the course; their grades ranged from A to D+. Interviewer 1 and Interviewer 2 were the third and second authors of this paper, respectively. All interviews were videorecorded and these recordings were transcribed. In addition, students' written work was retained. These videos, transcripts, and paperwork form the data examined in this analysis.

The interview protocol was designed to elicit students' construal of similarity between topics in linear algebra and high-school algebra, and to examine how they generalize a concept (e.g., invertibility) from one context to another. Accordingly, the topics covered by the interview were fairly wide-ranging, including one-to-one, onto, inverses, and compositions. This analysis draws on students' responses to the following questions that comprised approximately half of the interview protocol:

- Find the inverse of $f(x) = 3x - 9$.
- Find the inverse of $\mathrm{T}(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\mathbf{x}$.
- Why is it that we call both of these inverses?
- What, if anything, in your mind is common about the process of finding the inverse?
- What in your mind is common about the inverse itself in each case?
- What in your mind is common about the composition of f with its inverse and T with its inverse?
- A set of questions about composition, both in the high-school algebra context and the linear algebra context. For example, find $g(f(x))$ where $f(x) = x^2 + 1$ and $g(x) = -x$ and give a geometric interpretation of the composition (these served to refresh students' memory of how to compose functions).
- What will you get when you compose $f(x)$ with its inverse that you found earlier?
  - Perform the composition. Does the result match your prediction? If not, is there some reason your result makes sense?
- What will you get when you compose $\mathrm{T}(\mathbf{x})$ with its inverse?
  - Perform the composition. Does the result match your prediction? If not, is there some reason your result makes sense?

To analyze the data, we employed grounded theory (Strauss & Corbin, 1994). We made several passes through the data, identifying differences and patterns in student reasoning. As this analysis progressed, we were surprised to find that *all 10 students* predicted that the composition of a function with its inverse would yield 1. Six students (referred to as "resolvers") were able to resolve the discrepancy between their prediction and the correct answer they later obtained, while the other four ("non-resolvers") were not. That is, "resolvers" are precisely those students who were able to resolve the discrepancy during the interview, and "non-resolvers" are precisely those students who did not.

While we were somewhat dismayed that all ten students made an incorrect prediction, we were happy to see that at least some of the students were able to make sense of the correct answer they later obtained. This surprising result led to the following two research questions:

(1) What reasons do students give that the composition of a function or transformation with its inverse should be 1?
(2) What differentiates the mathematical reasoning of resolvers from that of non-resolvers?

4. Analysis

We found that the primary distinction between the reasoning of resolvers and non-resolvers was that resolvers exhibited patterns of reasoning that we call *do-nothing function* (DNF) reasoning. The hallmark of DNF reasoning is to express the result of the composition of a function with its inverse as the function whose output is the same as its input. In more formal, correspondence-based language, DNF reasoning might be that the identity function is a mapping between two copies of the same set in which each element is mapped to a copy of itself. Closely related, and often expressed in conjunction, is what we call *net do-nothing function* (net-DNF) reasoning. A student exhibiting net-DNF reasoning might say that the function does something which its inverse undoes, or that a vector is changed and then changed back to what it was.[1]

In the language of Sfard (1991), net-DNF reasoning is consistent with an operational (or process) view of composition of a function with its inverse, while DNF reasoning is consistent with a structural (or object) view. DNF reasoning might indicate that the "out-and-back" process expressed in net-DNF reasoning has been interiorized or condensed into a function object whose result is no change.

Once the categories of DNF and net-DNF reasoning emerged from our analysis, we coded each student's transcript for instances of these categories. Examples from our interviews of both DNF and net-DNF reasoning are presented in the case studies below.

DNF and net-DNF reasoning appeared to be particularly powerful ways for students to reason about the identity function, as well as to reconcile their incorrect prediction with their correct answer. We found that one pattern of reasoning by resolvers was that they used DNF and/or net-DNF reasoning, while a pattern of reasoning by non-resolvers was that they did not use DNF or net-DNF reasoning. This is to say, in our sample, each of the six resolvers exhibited DNF and/or net-DNF reasoning

[1] Note that several different types of metaphorical expressions are used in this description. We suggest that one reason DNF reasoning so useful for students is that they can be easily understood in each of several metaphors; for a further analysis of the clusters of metaphorical expressions used by the students in this study, see Zandieh et al. (2013).

when explaining the result they obtained when composing a function (or transformation) with its inverse, and none of the four non-resolvers ever exhibited DNF or net-DNF reasoning.

4.1. Prototypical non-resolver: Nila

We first consider the case of Nila, who we chose as a representative of the students who were unable to resolve the discrepancy between their prediction and their result. At the time of this interview, Nila was a sophomore majoring in mathematics. Nila was first asked to find the inverse of the function $f(x) = 3x - 9$, which she did quickly, correctly, and confidently. She hesitated only slightly when writing down the formal notation $f^{-1}(x)$, seeking confirmation that this is the proper notation to use.

Nila's next task was to find the inverse of the transformation $\mathrm{T}(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\mathbf{x}$. She performed the typical calculation to find the inverse of the matrix $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$, augmenting the matrix with the identity and row-reducing. Just like the previous calculation, this calculation did not appear to cause her any significant difficulty.

Nila was next asked to predict the result of composition of the function $f(x) = 3x - 9$ with its inverse, which she found earlier:

Nila: Oh, so you're saying if I put these together and then? Okay. Oh, in that case, if you take this one [points to $f(x) = 3x - 9$] and multiply it by this one [points to the inverse], it's supposed to give you 1 or is it −1? I forgot. I think it's 1, let me see.
Int. 1: So you're going to multiply them?
Nila: Yeah, I'm going to multiply them. Yeah, I think it's supposed to give me 1.

This is an example of the common confusion between multiplication and composition; this confusion was exhibited by all of the non-resolvers and none of the resolvers. However, as Nila further examined the expression, she became unconvinced that multiplication will give her the result she wanted, because as the interviewer pointed out, the result is "going to be x^2 and a bunch of ugly stuff."

Per the interview protocol, the interviewers proceeded through the questions on composition; she was able, for instance, to compose $f(x) = x^2 + 1$ with $g(x) = -x$. The interviewers then asked her again to predict the result of composition of f and its inverse. She repeated her initial prediction: "I think that would work out to be 1." The interviewers had her carry out the composition; she reached the correct answer, x, but seemed startled:

Int. 1: So you're surprised you got x instead of 1?
Nila: Um-m-m!
Int. 1: Or is that a good thing that you got x?
Nila: [Emphatically] I have no idea.
Int. 1: It is the right answer.
Nila: I don't know why I was thinking 1, but I was thinking 1.

The transcript here does not do justice to Nila's emotional expressions as documented in the video of her interview. While making the distressed noise here represented as "Um-m-m!", she moved her hands as if pushing away the offending paper, and the tone of her voice suggested hostility, as if the problem had tricked or betrayed her. She made no attempt to reconcile this result with her incorrect prediction, though she was clearly distressed by the contradiction and by her inability to see a resolution. The interviewers moved on to the next question without pressing her further. Nila's next task was to predict the result of composition of a transformation with its inverse. She predicted that the result would be the identity matrix, but did not carry out the calculation because the time allotted for the interview had elapsed.

4.2. Prototypical resolver: Jerry

Our analysis continues with Jerry, who we chose as a representative of the resolving group. At the time of this interview, Jerry was a senior in computer engineering, taking linear algebra as a major requirement. Jerry was first asked to find the inverse of the function $f(x) = 3x - 9$. At first, he was unable to remember how to do it, so the interviewers reminded him of the common algorithm, to interchange x and y and solve for y. (Reminding students of the procedure, if necessary, was common practice across all the interviews, as the interviews were not focused on whether a student remembered the procedure for finding the inverse.) Even after this reminder, Jerry seemed hesitant ("I don't really want to try"), but with a little more encouragement from the interviewers, he produced the correct answer without further difficulty.

Next, Jerry was asked to find the inverse of the transformation $\mathrm{T}(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\mathbf{x}$. As he began writing down the matrix, Jerry said, "I don't know what the **x** is there, should I just block it?" The interviewer responded, "I guess, for now," in a somewhat questioning tone. By "block it," Jerry apparently meant to ignore it for the purpose of this calculation: he wrote

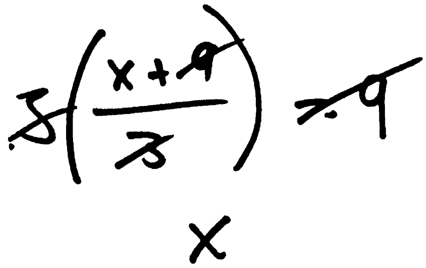

Fig. 1. Jerry's composition of f with f^{-1}.

down the matrix $\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ and augmented it with the identity matrix, and did not write the **x** anywhere. Jerry proceeded to correctly find the inverse of the matrix, row-reducing the augmented matrix and reading the inverse matrix off the right-hand side.

Jerry was then asked to predict the result of the composition of $f(x) = 3x - 9$ and its inverse which he found earlier. His initial prediction was 1, because "they sort of cancel each other." He was then asked to carry out the calculation, pictured below in Fig. 1, and produced the correct answer, x.

The interviewers pressed for his thinking:

Int. 1: Does that surprise you?
Jerry: The whole cancelation thing doesn't surprise me, but my original thought was . . .
Int. 1: 1? Do you think it's weird that it should be x when your initial guess was 1, or is there some reason why it makes sense for it to be x?
Jerry: Well, hm. No, whatever you put into it, that's what you're going to get out of it. This I'm thinking about with the x, whatever x you have, put into the function. And then run it again with its inverse, you're pretty much just going back to x.

Here, Jerry interpreted the function $f(f^{-1}(x)) = x$ with DNF reasoning: "whatever you put into it, that's what you're going to get out of it." Jerry's interpretation of the result of the composition with DNF reasoning came from the use of net-DNF reasoning: since the function does something and the inverse undoes it, "you're pretty much just going back to x." Taken as a whole, the composition does nothing. We noted this pattern (expressing the composition of f with f^{-1} as the DNF, then explaining why with net-DNF) in several of the resolving students. It appears to have been helpful for Jerry and the other resolvers to leverage both net-DNF reasoning of something being done and then undone and DNF reasoning about the composition as a whole.

Near the end of the interview, the interviewers revisited the question comparing the prediction with the result. Jerry again used DNF reasoning to explain how he came to understand the result he obtained:

Int. 2: Any final thoughts on your original prediction for f composed of f inverse to equal 1?
Jerry: How 1 would work out? I just sort of saw it as canceling, just a bunch of canceling each other out, you end up with just 1 by it. Uh.
Int. 2: Is the canceling like f and f to the negative 1st, like f over f, do those cancel, is that what's canceling to give you 1? Or is it something else canceling?
Jerry: Yeah, I see the functions canceling. But the, I don't know, now it just makes more sense that's whatever you put in there, is whatever you're getting out.

Jerry explained here that he *originally* saw the function and its inverse as canceling to yield 1. *Then*, however, he decided that x is a more reasonable answer, again because "whatever you put in there, is whatever you're getting out." Jerry thus appears able to view the result of the composition process (i.e., x) as a function in its own right, and to view this function as the function that does nothing.

Now that Jerry had used DNF and net-DNF reasoning to discuss the composition $f(f^{-1}(x)) = x$, he was able to draw an analogy with linear transformations, and to make a correct prediction about the composition of a transformation and its inverse. He explained his prediction in net-DNF terms:

Int. 1: If you compose T and T inverse, so similar to this but T's, T inverse, what would you predict that you'll get?
Jerry: I, what's a, just **x** again.
Int. 1: You think you might get **x** again, how come? Just because you have **x** here, or some other reason?
Jerry: We kind of did this in class. You're pretty much transforming it into something else, and the inverse really just transforming to, or transforming it back to what it originally was.

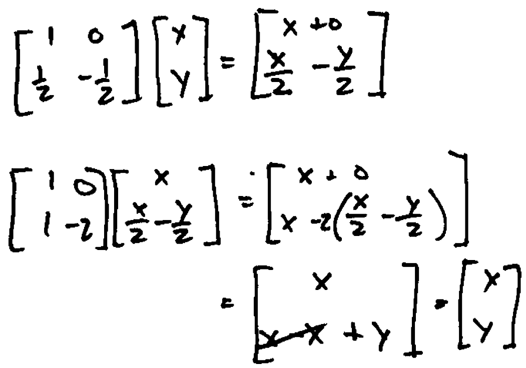

Fig. 2. Jerry's calculation of $T(T^{-1}(\mathbf{x}))$.

Jerry's language here echoed his prior language: "transforming it into something else, and ... transforming it back to what it originally was" is quite similar to "put[ting *x*] into the function ... then run[ning] it again with its inverse, you're pretty much going back to *x*." In this way we see that Jerry was able to use net-DNF reasoning and DNF reasoning to reason about this problem in *both* the function and the linear transformation contexts.

Jerry was then asked to compute the composition of T and its inverse, and compare the result with his prediction. The way in which he carried out this computation is reflective of the reasoning discussed above. Instead of computing the composition by multiplying the two matrices first, Jerry worked through the calculation by successively applying the two transformations to an arbitrary vector, as depicted in Fig. 2. This way of calculating the composition is perhaps more labor-intensive, but for Jerry, it appeared to both reflect and confirm how he had been reasoning about the result of composition of a transformation and its inverse.

Jerry was satisfied with this result; when asked if this was what he expected to get, he replied in the affirmative.

In both the function case and the transformation case, Jerry concluded, when you compose with the inverse, "you end up with the same input." It seems clear that the do-nothing function has provided him with a useful way to think about composition with the inverse. With DNF reasoning as a lever, Jerry appeared to condense the composition of the two functions (or transformations) into one new function (or transformation), the DNF, for which the output is the same as the input.

4.3. Transition: Lawson

Finally, we present the case of Lawson, a senior in computer science at the time of the interview. We chose this case because Lawson made a transition, with minimal intervention of the interviewers, from non-resolving to resolving: that is, he went from being unable to resolve the discrepancy between his prediction and his result to producing a sensible explanation of the discrepancy. We counted Lawson among the six students who resolved.

After seeing the ways of thinking exhibited by Jerry, and the similar thought processes of other students who were resolvers, the interviewers conjectured that Jerry's approach of successively applying the two transformations to an arbitrary vector might be particularly insightful for non-resolvers. Near the end of Lawson's interview, and in the spirit of a one-on-one teaching experiment, the interviewers followed up on this conjecture with a brief intervention designed to leverage what the interviewers were learning from the resolvers in order to help Lawson come to a resolution. The interview with Lawson was the last of the ten interviews, and was the only one in which the interviewers attempted to influence students' reasoning; accordingly, Lawson was the only student who transitioned from non-resolving to resolving, because his interview was the only one in which the interviewers pressed toward a resolution.

Lawson's interview proceeded similarly to Jerry's and Nila's. Although he had expressed earlier in the interview a confusion between the multiplicative inverse (i.e., the reciprocal) and the functional inverse of a function, he applied the standard procedure of switching *x* and *y* and solving for *y* (without being prompted by the interviewer) when asked to find the inverse of $f(x) = 3x - 9$, and found the inverse correctly. He explained that his confusion between these two inverses stemmed from the common notation used to represent both: "I just remember the inverse notation being this [a superscript -1], and I think I just automatically applied that for some reason."

Lawson was next asked to find the inverse of the transformation $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \mathbf{x}$. He made a false start by augmenting the matrix with the zero vector rather than the identity matrix, but quickly realized and corrected his mistake. The rest of his calculation proceeded quickly and he accurately found the inverse of the matrix.

When asked to think in general about the composition of a function or a transformation with its inverse, Lawson said, "whenever I see something like this [pointing at $f(f^{-1}(x))$], I automatically just want to cancel them out, make them 1 or something." This provides further evidence of Lawson's conflation of multiplicative and functional inverses: in the world of multiplicative inverses, an object and its inverse do indeed cancel to give 1.

$$y = \cancel{3}\left(\frac{x+\cancel{9}}{\cancel{3}}\right)\cancel{-9}$$
$$y = x$$

Fig. 3. Lawson's composition of f with f^{-1}.

Lawson reiterated this prediction for the result of the composition of f with its inverse, but when he carried out the calculation, he correctly obtained $y = x$ (see Fig. 3).

He did not appear to be as surprised or shocked as Nila, but still could not see a way to reconcile this answer with his prediction:

Int. 1: Does it surprise you that you get $y = x$?
Lawson: It doesn't surprise me, I guess, I'm not really fresh in mathematics, I would say. Linear algebra doesn't really take me back to anything I learned in the past, and I haven't done any normal algebra for a long time, so.
Int. 1: See, you initially predicted it would be 1, it turns out to be an x; do you have any way of thinking about why it's x instead of 1?
Lawson: [thinks] No.

He spent some time thinking about it, but was unable to make any progress, so the interviewers moved on to the next question. Thus far, his interview was similar to those of the non-resolvers: the x versus 1 discrepancy was encountered, the student made unsuccessful attempts to resolve it, and the interviewers progressed with the rest of the interview.

Lawson was next asked to predict the result of composition of T with T^{-1}:

Lawson: I assume it's going to equal the identity.
Int. 1: Okay, so let's see if it does. Was there a reason you assumed it was going to be the identity?
Lawson: I think originally, because I thought of reciprocals. When I tried to figure it out this way originally, I thought it was like this [writes A/A].

Once again, Lawson exhibited a conflation of the different notions of inverse; here, as evidenced by his division notation, he appeared to confuse the multiplicative inverse (i.e., the reciprocal) with the inverse of a matrix. He then carried out the calculation (see Fig. 4); unlike Jerry, and without net-DNF reasoning to spur him on, he computed the composition by multiplying the two matrices. His result, the identity matrix, was what he expected.

After they had completed the interview protocol, the interviewers took the opportunity to test their conjecture that Jerry's method of computing the composition might be helpful for students in coming to understand the result of the composition. To foreground the functional nature of the linear transformations, Interviewer 2 began by pointing out an important difference between the way Lawson symbolized the two problems:

Int. 2: When you did f composed with f inverse, the input variable x was always present. When you did the T composed T inverse, the input for transformations wasn't present.
Lawson: [nods]
Int. 2: So that seems to be an important difference. So I'm wondering what in your mind is the role of the input in the T composed T inverse? And how can you think about the role of the input, the things that you input into transformations, as you're thinking about computing T, T inverse?
Lawson: In this case [functions], I'm plugging this [circles his expression for $f^{-1}(x)$] into where x was [points at $f(x)$], because x is present. Whereas in this case [transformations], it's not present. I'm not sure what you mean by 'not present' necessarily because we have $\mathbf{x}$ here [underlines second $\mathbf{x}$ in $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \mathbf{x}$].
Int. 1: It's there, but it's not here [points to matrix computation in Fig. 4]. When you wanted to find out what this is, it didn't appear any more.

$$\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Fig. 4. Lawson's multiplication computation for composition.

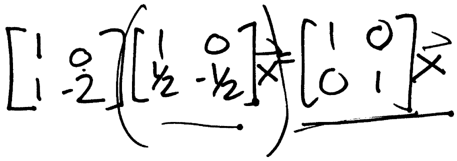

Fig. 5. Lawson adds **x**'s to his calculation.

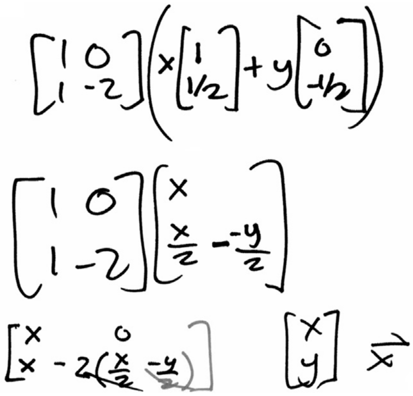

Fig. 6. Lawson does what Jerry did.

As a first attempt to help Lawson reconcile this difference, Interviewer 1 had Lawson generate an expression for $\mathrm{T}^{-1}(\mathbf{x})$ parallel to the one printed on the paper for $\mathrm{T}(\mathbf{x})$:

Int. 1: Here's a question for you. Write for me here $\mathrm{T}^{-1}(\mathbf{x})$ equals, now fill in the blank. Like how here we have $\mathrm{T}(\mathbf{x})$ equals something, so we want $\mathrm{T}^{-1}(\mathbf{x})$ equals something.
Lawson: Oh, okay. You could note it [writes $\mathrm{T}^{-1}(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$; pauses]. And I would say you can put the **x** here [writes in **x** after the matrix], I guess.
Int. 1: So what happens if I have the **x** there, does that change what's on the right side?
Lawson: [adds **x** to the end of the matrix calculation; see Fig. 5] I would assume it has an **x** there. So that's the identity times **x**. Which, will that come out as the $y = x$ equation?

Even after writing this expression and making this observation, Lawson appears unconvinced, using a questioning tone in the above portion of transcript and agreeing with Interviewer 2's observation that his difficulties did not seem to be resolved yet. He said he wished that he had "some kind of revelation," but clearly did not.

Interviewer 2 then took another tack in his efforts to help Lawson recognize the functional nature of the linear transformation. Since the arbitrary-vector version of the composition calculation had been helpful for Jerry, Interviewer 2 reasoned that being more explicit about this might be equally helpful for Lawson.

Int. 2: So what does this mean to say $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ times the vector **x**? Well, it means 1 times the 1st component plus, etc., right?
Lawson: Um-hm.
Int. 2: So if you wrote that out in terms of the symbols x and y, then you would be able to say, 'Now I need to use that and get acted on by the external function.' So I'm curious, could you just push the notation that way a bit?

With this impetus from the interviewers, Lawson proceeded through a calculation similar to the one performed by Jerry, summarized below in Fig. 6.

Lawson: You end up with x, y. This cancels, x would cancel, and this [y] would be positive.
Int. 2: So going back to what you were originally computing, was T composed T inverse of the vector x, y. And this equals?

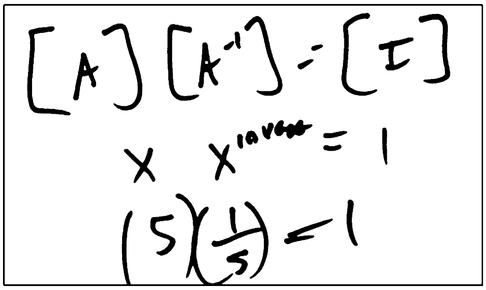

Fig. 7. Nigel writes three different inverses.

Lawson: Essentially this [the vector (x, y)] is the vector $\mathbf{x}$, so essentially I did end up with, when I composed them, I ended up with $\mathbf{x}$ as in the, whatever I had here. Yeah, it is identical [to the function case]. That's cool! [Laughs] I'm glad I did that, that's interesting.

We note that Lawson's resolution, and his sudden feeling that the process made sense, came at the very moment he expressed DNF reasoning: "I ended up with $\mathbf{x}$ as in the, whatever I had here," i.e., the vector (x, y) that he started with. His DNF reasoning resulted from a calculation that illustrates net-DNF reasoning. This is further evidence of the utility and power of these ideas in students' explanations.

5. Discussion

These case studies both illustrate the utility of DNF and net-DNF reasoning in students' reconciliation processes, and illustrate two reasons students predict that $f(f^{-1}(x))$ should be 1. The first of these is the fairly common conflation of functional inverses with multiplicative inverses. The second is the influence of new knowledge on prior knowledge. To help researchers discuss such influence, Hohensee (2011) borrowed from the study of language learning the notion of *backward transfer*: how prior knowledge changes as new knowledge is built upon it. This notion extends the traditional account of transfer as the influence of prior activities upon new situations. It should be noted that backward transfer, as defined by Hohensee and used here, is a value-neutral term; it is equally possible for new knowledge to positively or negatively impact students' prior knowledge. In this section, each of these themes will be discussed in greater depth, and we will explore some reasons why we think DNF and net-DNF reasoning were so productive for these students.

5.1. Conflation with multiplicative inverses

In the previous analysis, we alluded to several examples of conflation of the various mathematical objects all called "inverse" and all symbolized with a superscript -1. In particular, students participating in this interview discussed three distinct mathematical objects all symbolized this way: the multiplicative inverse of a number (i.e., the reciprocal), the functional inverse, and the multiplicative inverse of a matrix. Nigel, for instance, said: "So say you have x, the inverse is x to the negative 1, or 1 over x." He later wrote down two other multiplicative inverses as shown in Fig. 7.

To help reduce the confusion between these concepts, teachers might explicitly address the similarities and differences between them. Linear algebra is an opportune place for this discussion to occur, as it is typically students' first exposure to more general fields where the "multiplicative" inverse is analogous to, but different from, the reciprocal in the field of real numbers.

Additionally, students often conflate composition with multiplication; for instance, Nila demonstrated this confusion before being reminded how to compose functions. This is not particularly surprising, especially given that the composition notation $f \bigcirc g$ even looks similar to multiplication. However, we propose an additional explanation: this may be due to the influence of linear algebra. The composition of two transformations is computed by multiplying matrices; this may lead students to attempt to compose functions by multiplying them. This would be one instance of backward transfer, but there are other possible instances.

5.2. Backward transfer

When asked to predict the result of composition of f with f^{-1}, some students seemed to know the right answer but simply symbolized it incorrectly. For instance, Gabe offered the following explanation of what the result should be:

Int. 1: If I do f of f inverse of x, what do you expect it to come out with?
Gabe: Input, the input that you put in there. It shouldn't modify it.
Int. 1: If I haven't put in any input though, I'm just doing a calculation?

Gabe: [writes] It's just 1.
Int. 1: It would be 1?
Gabe: It's not going to change what you put in there, because if you do something and then you undo it, has it really changed? It's like philosophy right there, it's going to be the same number in terms of, put in a 5, you're going to get out a 5.

Gabe appeared to know that the right answer is the do-nothing function: he explained that the composition "shouldn't modify [the input]" and that "it's not going to change what you put in there." He further illustrated this understanding with a specific example: "put in a 5, you're going to get out a 5." Accordingly, we propose that the only reason he didn't predict x, as we may otherwise have expected him to, is because he chose the wrong notation.

Why would Gabe and other students symbolize the do-nothing function incorrectly? This may be another instance of backward transfer from the symbolism of linear algebra. It is common practice in linear algebra classrooms to omit the $\mathbf{x}$ and work directly with matrices when performing calculations on linear transformations (a part-whole metonymy[2] in which the matrix represents the transformation). Thus, students are likely used to seeing the identity *matrix* alone when the identity *transformation* is under discussion. From here, it is no great leap to imagine students thinking that 1 is the identity function in the context of high-school algebra. Here, Gabe exhibited this metonymy, speaking of T (the name of a transformation) as if it were a matrix:

Gabe: If you get T, and you multiply T by its inverse, you should get the identity matrix, which is essentially 1.
Int. 1: So you see those as the same?
Gabe: Yeah. 1 in matrix algebra looks like this [writes an identity matrix], same thing.

To lend further support to this hypothesis, several students (including Gabe) reconciled their prediction with their result by explaining that x is the same thing as 1 times x. This is directly analogous to the notation used to represent the identity transformation: $\mathrm{T}(\mathbf{x}) = \mathrm{I}\mathbf{x}$.

5.3. *DNF and net-DNF reasoning*

As mentioned earlier, all six of the students who resolved, and none of the four who did not, used DNF and net-DNF reasoning in their explanation. Why were these ideas so useful for students? We return to the common pattern exhibited by several resolvers, including Jerry: he expressed the composition of f with f^{-1} as the DNF, and then explained why using net-DNF reasoning. Jerry's explanation was rooted in the process of pushing an arbitrary element through the function and its inverse and observing that the net result is no change. We conjecture that Jerry passed through the sequence of steps proposed by Sfard (1991): pushing an arbitrary element through his calculation allowed interiorization; reasoning about "putting in and going back" encouraged condensation.

Consider further the interviewers' intervention that led to Lawson's resolution. They asked Lawson to write the transformation as a matrix with an $\mathbf{x}$ attached, then to push an arbitrary element through the calculation. Once he had done this computation, the interviewers encouraged him to reflect on the result, and he appeared to have a sudden epiphany. Pushing an arbitrary element through the calculation encouraged interiorization: Lawson was attending to the specific details of the calculation. Asking him to reflect on the result encouraged condensation, since he could now ignore the fine details of the calculation and instead focus on the big-picture relationship between the beginning and ending state. Then, if interiorization and condensation did indeed occur, reification would become possible.

We conjecture that this pattern of reasoning can allow students to come to see the result of composition of a function (or transformation) and its inverse as a function (or transformation) in its own right, and reason that since it does nothing to an arbitrary element, it must be the do-nothing function. Further, we conjecture that this pattern of reasoning may strengthen students' understanding of the functional nature of linear transformations.

6. Pedagogical implications

Educators certainly do not want students leaving an undergraduate linear algebra class saying that $f(f^{-1}(x)) = 1$. How can this be avoided? One suggestion that emerges from our data is to pay more explicit attention to the common metonymy mentioned above of using matrices to stand for linear transformations, and omitting the $\mathbf{x}$ when doing computations with linear transformations. For those well-versed in the field, this is a practice that contributes to efficiency and fluency of computations. We view the ability to fluently and seamlessly suppress notation in this way as a mark of mathematical sophistication and efficiency, and we feel that students leaving a linear algebra course should feel comfortable doing this.

However, we suggest that for students, the metonymic meaning behind this practice may not be present, which might be problematic for their understanding of linear transformations. Thinking of a matrix *as* a transformation, rather than *representing* or *standing for* a transformation, may keep students from seeing that the transformation is indeed a function.

[2] For more examples and discussion of the use of metonymy by undergraduate mathematics students, see Zandieh and Knapp (2006).

This sort of false metonymy may be indicative of, or may contribute to the development of, a pseudostructural conception (Sfard, 1992) of linear transformation: the student might see the matrix *as* the function object, without being able to unpack this object to reveal the functional process (i.e., multiplying a vector by the matrix). This is similar to the classic example of a pseudostructural conception of function described by Sfard (1992): “a formula as a thing in itself, not standing for anything else” (p. 75). For a student with a pseudostructural conception, the meaning of multiplying a vector by this matrix (or, indeed, that vectors should be involved at all) would be unclear. This sort of pseudostructural understanding may in turn hinder their understanding of many further topics in linear algebra which require a functional understanding of linear transformations.

In order to build the true metonymic meaning of using a matrix to stand for a transformation, students need explicit practice working with transformations as functions, with the input and output foregrounded; this is the essence of the intervention that helped Lawson resolve. Students would likely benefit from explicit attention to the use of the matrix with the vector rather than working with the matrices with the vector being implied only. Additionally, when a student speaks of a matrix as a transformation, it may not be clear to the instructor whether or not the metonymic meaning is present; we encourage instructors to probe students' understanding when linear transformations are first encountered, and until it becomes “taken as shared” (Cobb and Yackel, 1996) by the classroom community. We call for empirical attention to this issue in classrooms that are attentive to it.

In conversations with the instructor of these students, she indicated that in classroom work, she made a point of paying explicit attention to functional aspects of linear transformation, as well as introducing DNF reasoning to the students. Perhaps her attention to these details enabled the six resolvers to successfully grapple with the problem posed in these interviews; we take this as a good sign for the practicability of the teaching recommendations we propose.

It appears that DNF and net-DNF reasoning is a particularly useful way to characterize the identity function; indeed, this may even be the most useful or most “correct” way. Calling the identity function “the function that does nothing” (as the instructor of this class did) helps students make the link between the composition of a function with its inverse and the symbols $f(f^{-1}(x)) = x$ or $(f \bigcirc f^{-1})(x) = x$ that represent it. Additionally, we see DNF and net-DNF reasoning as indicative of a sophisticated understanding of the identity function; in particular, we think this demonstrates a deeper understanding than simply knowing the name “the identity function.”

We further suggest that instructors might follow the example of the instructor of the students in this study by using DNF and net-DNF reasoning to help students develop an intuitive grasp of the identity concept. We call attention again to the interviewers' intervention that led to Lawson's resolution, discussed above. We suggest that since pushing an arbitrary element through a composition calculation foregrounds the functional nature of the linear transformation, assigning a problem of this type to linear algebra students can be a powerful learning opportunity. Future work might examine the ways of thinking that a wide range of students bring to bear on such a problem, and whether this intervention would be as successful with other students, or in a full-class setting.

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