

A hypothetical learning trajectory for conceptualizing matrices as linear transformations

Christine Andrews-Larson, Megan Wawro & Michelle Zandieh

To cite this article: Christine Andrews-Larson, Megan Wawro & Michelle Zandieh (2017) A hypothetical learning trajectory for conceptualizing matrices as linear transformations, International Journal of Mathematical Education in Science and Technology, 48:6, 809-829, DOI: [10.1080/0020739X.2016.1276225](https://doi.org/10.1080/0020739X.2016.1276225)

To link to this article: <http://dx.doi.org/10.1080/0020739X.2016.1276225>



Published online: 01 Feb 2017.



Submit your article to this journal [↗](#)



Article views: 35



View related articles [↗](#)



View Crossmark data [↗](#)



A hypothetical learning trajectory for conceptualizing matrices as linear transformations

Christine Andrews-Larson^a, Megan Wawro^b and Michelle Zandieh^c

^aSchool of Teacher Education, The Florida State University College of Education, Tallahassee, FL, USA;

^bDepartment of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA, USA;

^cDepartment of Applied Sciences and Mathematics, College of Technology and Innovation, Arizona State University, Mesa, AZ, USA

ABSTRACT

In this paper, we present a hypothetical learning trajectory (HLT) aimed at supporting students in developing flexible ways of reasoning about matrices as linear transformations in the context of introductory linear algebra. In our HLT, we highlight the integral role of the instructor in this development. Our HLT is based on the 'Italicizing N' task sequence, in which students work to generate, compose, and invert matrices that correspond to geometric transformations specified within the problem context. In particular, we describe the ways in which the students develop local transformation views of matrix multiplication (focused on individual mappings of input vectors to output vectors) and extend these local views to more global views in which matrices are conceptualized in terms of how they transform a space in a coordinated way.

ARTICLE HISTORY

Received 5 February 2016

KEYWORDS

Linear algebra; hypothetical learning trajectory; linear transformation

1. Introduction

Hypothetical learning trajectories (HLTs) have been helpful for bridging the work of researchers and practitioners; the construct supports the content-specific documentation of common milestones and learning environments that support students' progression across those milestones. Although examples of HLTs exist in the research literature [1,2], there are few examples in undergraduate mathematics [3] and none that we know of in undergraduate linear algebra. This paper offers a pragmatic contribution in basic research on undergraduate teaching and learning of linear algebra by describing an HLT in undergraduate linear algebra, particularly with regard to supporting students' thinking about matrices as linear transformations.

An HLT serves the role of outlining a path that students can take to learning new content. We see our work as documenting a path to learning important ideas in linear algebra relating to linear transformations. Part of documenting this path (HLT) is the role that the teacher plays in this development. In this way, answering questions about the development of student thinking and the role of the teacher creates a possible path, an existence proof

of a path for student learning. Having such a model can aid other researchers, curriculum developers, and teachers. For researchers, it illustrates linear algebra understandings that are possible for beginning mathematics and science students, and adds to the body of knowledge documenting such understandings. For curriculum developers, it describes potential paths for curriculum design. For teachers, it provides suggestions as to ways to draw out their students' understandings in this content area and perhaps more broadly.

In this paper, we specifically examine the following research questions in the context of one classroom implementation of this HLT:

- (1) What is the nature of student reasoning elicited, and how did that reasoning shift across a task sequence designed to support student learning about matrices as linear transformations in an introductory undergraduate linear algebra course?
- (2) What moves and decisions did the instructor make in implementing this sequence of tasks, and how did these relate to the trajectory of student reasoning?

In answering these questions, we characterize the development of student thinking about linear transformations and how that development can be supported instructionally. In addition, we draw on and extend literature on student understanding of functions and linear transformations [4–6].

There is a growing body of work within mathematics education design research that honours a tight integration of theory and practice. On one hand, research into student thinking in mathematics informs the creation and enactment of instructional materials; on the other hand, analysing the implementation of these instructional materials informs theory about how students learn mathematics, both in content-specific ways and in general. The design research approach of classroom teaching experiments [7], which is central to our work in linear algebra, is especially well suited to this integration. The products of classroom teaching experiments often include both theoretical advances [8] as well as empirically grounded instructional innovations [3,9,10]. One such research-based construct that directly informs instructional practice is that of a hypothetical learning trajectory [1,2,11,12]. As described by Simon [2], an HLT consists of the following three components: learning goals, learning activities, and hypothetical learning processes. In this paper we present an HLT designed to support students in developing an understanding of matrices as linear transformations. In particular, we examine the evolution of students' mathematical reasoning in the context of a specific task sequence, and we explicitly attend to the role of the instructor in supporting the mathematical development that takes place in the classroom enactment of this HLT.

To situate this work, we first present a discussion of theory and literature that informed our work in developing these instructional materials to promote students' understanding of linear transformations. Next, we organize the results section in terms of the instructional tasks in which students engage, using the HLT's learning goals to provide a broad orientation to the section. For each task, we discuss what the students are being asked to do and how this is significant in terms of our learning goals. To address our two research questions, we use data from a classroom teaching experiment [7] in linear algebra to describe students' mathematical activity as they engage in each task, and the role of the instructor in supporting students in meeting the learning goals of the task sequence.

2. Literature review

The challenges students may face when learning fundamental concepts in linear algebra are well documented [13–20]. Linear algebra is a challenging course for students for a variety of reasons, including: generalizing geometric ways of reasoning [17,21–23]; understanding the symbolic notation of linear algebra [24,25]; working with formal definitions in linear algebra [15]; and coordinating across the different modes or representations involved in linear algebra [17,18].

The Linear Algebra Curriculum Study Group (LACSG) recommends that a first course in linear algebra be matrix-based and focus on a geometrically motivated development of \mathbb{R}^n prior to a formal treatment of vector spaces [26]. In accordance with these recommendations, our research-based instructional sequence aims to support students in developing flexible and productive ways of conceptualizing matrices as linear transformations. In this sequence we limit the focus to matrix representations of linear transformations from \mathbb{R}^2 to \mathbb{R}^2 that make use of the standard basis for both domain and codomain.

Three common interpretations of a matrix times a vector that are important for supporting student understanding of key ideas in introductory linear algebra are: linear combination interpretation, systems interpretation, and transformations interpretation [25]. In a linear combination interpretation, the multiplication is viewed as a linear combination of the column vectors of the matrix. A systems view of the matrix multiplication is typified by an effort to reinterpret matrix multiplication as corresponding to a system of equations. In a transformations interpretation, one conceives of the matrix as something that transforms one vector into another vector. The HLT we detail offers a means by which instructors can support students in developing a transformation interpretation of a matrix times a vector to a more global view of how a linear transformation defined by a matrix affects an entire space and how transformations can be composed.

Symbolization of algebraic ideas relies heavily on the use of variables and functions, and research shows that students at the undergraduate level struggle in their interpretations of functions [5,27,28]. Students tend to confuse functions with equations and formulas, and exhibit difficulty in making sense of function notation [27,5]. Oehrtman et al. [5] identify the ability to conceive of ‘functions as general processes that accept input and produce output’ (p. 154) as an essential understanding for conceptualizing both functions and their inverses. We posit that students’ difficulties with functions are amplified in the realm of linear algebra, where students must reckon with symbolization in multidimensional contexts. The HLT we lay out in this paper has the potential to support students in developing productive ways of thinking about linear transformations as examples of functions in the context of introductory linear algebra.

3. Theoretical framework

This development of the HLT in this paper draws on three instructional design heuristics of Realistic Mathematics Education (RME) [29,30]. First, an instructional sequence should be based on experientially real starting points. In other words, tasks that comprise an instructional sequence should be set in a context that enables students to immediately engage and make some initial mathematical progress. Second, the task sequence should be designed to support students in progressing toward a set of mathematical learning goals associated with the instructional sequence. Third, classroom activity should be structured

so as to support students in developing models-of their mathematical activity that can then be used as models-for subsequent mathematical activity [31]. In other words, the process of students' reasoning on a task becomes reified so that the outcome of that process of reasoning can serve as a meaningful basis and starting point for students' reasoning on subsequent tasks [32].

In order to operationalize these RME heuristics into content-specific deliverables explicitly related to instruction, a number of researchers have used the construct of a hypothetical learning trajectory. The construct of an HLT was initially developed for and has primarily been used by both instructors and researchers with a focus on individual student learning in particular content domains [2,11,33,34]. In addition to this use, the construct has been adapted by some to conjecture about the mathematical progress of a classroom as a whole [1,35,36]. We hold the view that, in a classroom setting, individual student thinking shapes and is shaped by the mathematical progress of the classroom community [37]. Indeed, Cobb et al. [35] describe an HLT as 'consisting of conjectures about the collective mathematical development of the classroom community' (p. 117), and Gravemeijer et al. [1] describe it as a 'possible taken-as-shared learning route for the classroom community' (p. 52). We follow this adaptation of the construct for the social perspective, adding here the explicit consideration of the role of the instructor as an integral aspect in the sense making that takes place in the classroom enactment.

In elaborating our HLT, we consider an HLT to be a storyline about teaching and learning that occurs over an extended period of time. The storyline includes four interrelated aspects:

- (1) Learning goals about student reasoning;
- (2) A sequence of instructional tasks in which students engage;
- (3) Evolution of students' mathematical activity;
- (4) The role of the instructor in supporting students' mathematical development across the sequence of tasks.

Note that the first three aspects we consider in this paper are consistent with Simon's [2] definition of HLT, whereas the fourth aspect reflects our attention to the role of the instructor. Our definition of an HLT differs from others in its explicit inclusion of the role of the instructor as integral in the anticipated progression of mathematical activity in the classroom. The instructor plays a unique role in the classroom community with responsibilities of pushing forward the development of mathematics and fostering productive social and socio-mathematical norms [37] within that classroom community. Thus, our framing highlights the multi-dimensional structure of classroom activity. As the first and third aspects highlight, the instructor's role includes considering the learning goals she has for her classroom, as well as envisioning the evolution of students' mathematical activity toward these goals through classroom activity. The second and fourth aspects of an HLT – the sequence of instructional tasks in which the students engage and the role of an instructor – speak to how these tasks are implemented in a given classroom.

4. Data sources and methods

This research-based HLT grows out of a larger design research project that explores ways of building on students' current ways of reasoning to help them develop more formal and

conventional ways of reasoning, particularly in linear algebra. The instructional sequence described in this paper was developed and iteratively refined over the course of four semester-long classroom teaching experiments [7] that took place in inquiry oriented introductory linear algebra classes at public universities in the southwestern United States. These classroom teaching experiments were conducted in classes ranging in size from 20-40 students, all of whom had been required to complete two semesters of calculus prior to enrolling in linear algebra. Most students in these classes were majoring in math, computer science, or engineering.

We use the term inquiry-oriented in a dual sense, where the term inquiry refers to the activity of the students as well as the instructor [38]. Students engage in discussions of mathematical ideas, questions, and problems with which they are unfamiliar and do not yet have ways of approaching; thus, evaluating arguments and considering alternative explanations are central aspects of student activity. Instructor activity includes facilitating these discussions, in which the instructor consistently inquires into students' mathematical reasoning. More specifically, we broadly align our approach to mathematics teaching and learning with the work of Freudenthal [29] and Gravemeijer [31]. Specifically, in viewing mathematics as an inherently human activity, we aim to structure students' mathematical learning around their work (much of which is done in groups with their peers) solving novel problems designed to recreate a need for important mathematical ideas. The approaches and conjectures students develop as they work on these tasks serve as the basis for mathematical discussions aimed at supporting students' learning of topics at hand and their reinvention of important mathematical ideas. In these settings, the role of the instructor is to select and pose appropriate mathematical tasks to students, provide them with opportunities to engage in solving these tasks, and structure classroom discussions following students' work on these tasks so that students have the opportunity to discover important mathematical relationships. The role of the instructor is to guide the discussion so that it attends to important mathematical ideas and to link the language and notation of students to more conventional notation of the mathematics community as the need for appropriate language and notation arises through careful design and sequencing of high quality mathematical tasks.

During each classroom teaching experiment, we videotaped every class period using three video cameras that focused on both whole class discussion and small group work. We also collected student written work from each class day. As a research team, we met approximately three times a week in order to debrief after class, discuss students' work and mathematical development in the class, and plan the following class. We also used these meetings to inform decisions regarding the subsequent iteration of the classroom teaching experiment. Over the four CTEs, as we deepened our knowledge of student thinking and its development in linear algebra, we concurrently refined the learning goals and instructional tasks to reflect what we learned about students' conceptual resources and sources of struggle. Importantly, we have come to better understand the instructors' role in supporting students' learning in this context. One of the results from this extensive iterative work is an articulation of an HLT for interpreting linear transformations in the context of matrix multiplication. Examples presented in this paper were taken from the fourth classroom teaching experiment, in which one of the authors of this paper served as teacher-researcher and her method of instruction aligned with the aforementioned characterization of inquiry-oriented instruction.

In order to develop and articulate the HLT presented in this paper, instructor and student notes were used to reconstruct the broad progression of classroom activity across the set of tasks; these were used to identify relevant segments of classroom video from the classroom teaching experiment. Our research team generated memos to document students' mathematical activity and the role of the instructor in progressing through this particular enactment of the instructional sequence, paying particular attention to the variety of student interpretations elicited by the task, issues that were problematic for students, and the role of the instructor in negotiating sense-making around student generated notation and connecting to more conventional notation used by the broader mathematical community.

Prior to the instructional sequence driven by this HLT, the class had engaged in an RME-inspired instructional sequence focused on conceptual understanding of linear combinations, span, and linear independence [39]. The class also developed solution techniques for linear systems to help answer questions regarding span and linear independence of sets of vectors. This led to the definition and exploration of equivalent systems, elementary row operations, matrices as arrays of column vectors, augmented matrices, Gaussian elimination, row-reduced echelon form, pivots, and existence and uniqueness of solutions. In summary, students were familiar with ideas about linear independence and span, as well as methods of solving systems of linear equations (including Gaussian elimination) and interpreting their solution sets.

5. Results

Our research questions focus on the evolution of students' reasoning and the teacher's role in this evolution. These questions are tightly linked with the last two components of our definition of an HLT (the evolution of students' mathematical activity and the role of the instructor in that evolution). As such, we use the first two components of an HLT (learning goals and sequence of tasks) to structure our results. The HLT developed in this report encompasses four learning goals: (a) interpreting a linear transformation as a mathematical entity that transforms input vectors to output vectors; (b) interpreting matrix multiplication in terms of the composition of linear transformations; (c) conceptualizing an inverse transformation as 'undoing' the original transformation; and (d) conceptualizing linear transformations as entities that geometrically transform a space. In our analysis, we noted that rather than being purely sequential in nature, these four learning goals interweave as students' reasoning develops. In particular, we view learning goal (a) as a local view of linear transformation wherein inputs are thought of as being transformed into outputs one at a time, whereas we view learning goal (d) as reflecting a more global view wherein the set of inputs is thought of as being transformed into the set of outputs in a coordinated way. The global view does not replace the local view, but rather elaborates it. In the following sections, we consider in particular the ways in which students develop and relate local and global views, as well as the role of the instructor in supporting this development.

The sequence of tasks that comprise the HLT begins with (1) an introduction to a transformation view of the matrix equation $A\mathbf{x} = \mathbf{b}$. After that, it is composed of three main tasks: (2) the Italicizing N task; (3) the Pat and Jamie task; and (4) the Getting Back to the N task. In the following sections, these tasks are illustrated as we address our research questions with data from the fourth classroom teaching experiment.

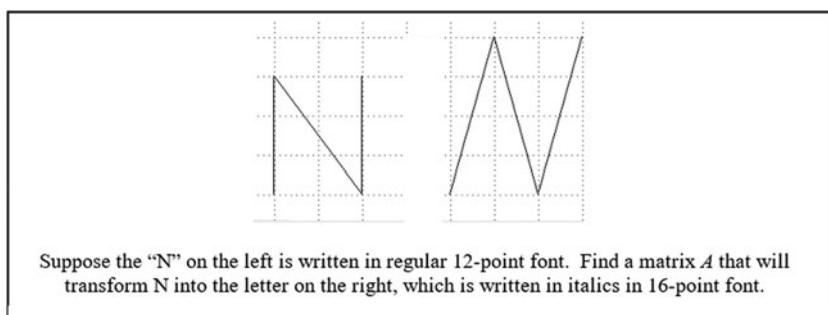


Figure 1. The Italicizing N task.

5.1. Setting the stage for Task 1: introducing a transformation view of $Ax = b$

Before introducing the first task, the instructor in the classroom teaching experiment introduced an interpretive shift toward a transformation view of $Ax = b$, using the equation $\begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \end{bmatrix}$ as an example. The instructor reminded students that they had previously developed a column view and a row view of this equation. The column view interprets $Ax = b$ as a linear combination of column vectors, which she indicated could be seen geometrically through tip-to-tail vector addition with 1 and 4 as weights on the column vectors of A . The row view interprets this equation as a system of linear equations, which the instructor pointed out could be seen geometrically as an intersection of lines with (1,4) as the point of intersection. Introducing the interpretive shift, the instructor told students "Today we want to think about a new way, where we think about A acting on x , kind of like an input, and b being the output. We're going to call this one the transformation view."

After introducing this shift in interpretation of the matrix equation $Ax = b$, the instructor related this interpretation to students' high school experiences with functions and provided students with definitions and examples of the terms transformation, domain, and codomain.

5.2. The Italicizing N task

In the Italicizing N task (see Figure 1), the students are asked to determine a matrix A that represents the requested transformation of the letter N, as described in the problem statement. The task, which was inspired by an example application in Lay's [40] linear algebra textbook, is the experientially real starting point for the HLT [31].

In our analysis, we identified three key issues that emerged in classroom discussion around students' work on the italicizing N task: (1) what the dimensions of the transformation matrix A should be, (2) how to label the pertinent information on the two graphs of the N, and (3) how many input-output pairs are needed to determine A . In our discussion of these issues, we foreground the role of the instructor by attending to the moves the instructor made that reflect decisions about how to use student thinking to address broader mathematical issues and about what to tell when. Indeed, the emergence of these three issues in classroom discussion is a reflection of both what issues emerged for students as they worked on the task, as well as what issues the instructor chose to emphasize in discussing students' work on the task.

5.2.1. Determining the dimensions of A : labelling inputs and outputs

The instructor in our CTE introduced this task at the end of the class period in which she had defined transformation, domain, and codomain, telling students, ‘You’re coming up with a matrix. The values of the matrix will matter. You need to think with your group about what size the matrix should be.’ After students had spent a few minutes discussing this in small groups, there was a brief whole class discussion in which one student shared his idea about labelling the left segment on the original N and the italicized N as vectors in \mathbb{R}^2 , which the instructor related to the previously developed language for domain and codomain. Before students left, each student filled out a reflection indicating his/her group’s progress on the Italicizing N task. Many students focused on their ideas about how to label inputs and outputs, but a number also commented on what they thought the dimensions of the matrix A might be; possibilities suggested included 3×1 , 2×2 , and 2×3 .

The instructor began the following class session with a conversation about domain and codomain for the task at hand, using students’ reflection responses as leverage towards telling them that A should be a 2×2 matrix:

Instructor: What about domain and codomain for this italicizing N ? Anyone remember that aspect of Monday?

Student: The domain is the input and the codomain is your output.

Instructor: Yeah. The domain is the input space, and the codomain is the output space. If you think about italicizing N , a lot of us think you can have vectors from \mathbb{R}^2 , and the output we were thinking of as vectors in \mathbb{R}^2 . So you’re saying that the transformation is taking us from \mathbb{R}^2 to \mathbb{R}^2 . I saw on a few of your papers that now we have this, so then what would A look like? ... A lot of you said we need it to be a 2×2 .

We note that the student’s comment in this exchange hinted at a local view, referring to the domain as ‘the input’ and the codomain as ‘your output’; the instructor pressed toward a more global view by rephrasing this contribution in terms of the input *space* and the output *space*. Following this exchange, the instructor sent students back to work with their groups to find the matrix that would yield the desired transformation before bringing students back together to share their approaches in whole class discussion. In the discussion that ensued, the students described their efforts to select a consistent labelling system for inputs and outputs. The following exchange shows how one student drew on a local view to explain why the transformation matrix should be 2×2 , and highlights two strategies for representing inputs and outputs: using free vectors in \mathbb{R}^2 or using points in the x - y plane.

Kyle: We figured out that it has to be a two by two matrix, so that when you multiply it out by a one by two [sic], you actually end up with a one by two [sic] matrix. So, we know that. But the question is what goes in here [points to empty input vector, Figure 2], and what goes over here? [Points to the empty output vector, Figure 2.]

Kyle: So if you’re looking at your N both ways [draws the original N and the tall italicized N as shown in Figure 2], then basically the way we had it set up is that this one looks like a set of three vectors. Zero, three since over zero up three here. And you have two, negative three, and you have two, three. Or no, zero, three ...Cody was it two, three?

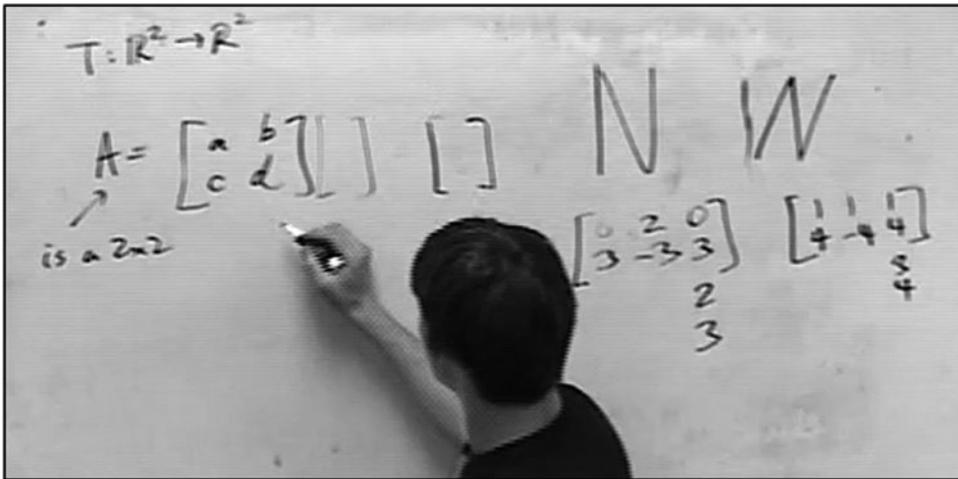


Figure 2. Kyle's explanation of his group's approach to Task 1.

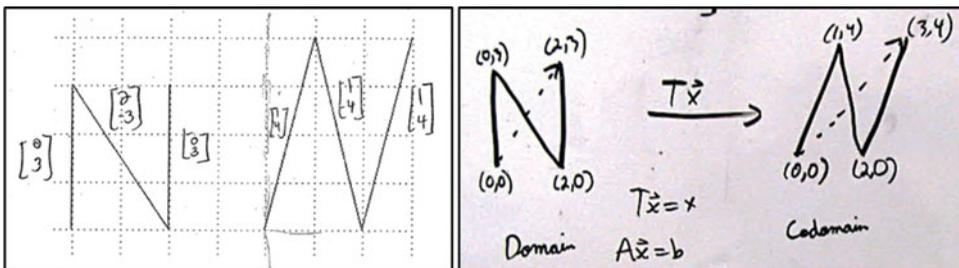


Figure 3. Examples of student work illustrating two ways of labelling.

Cody: I still think it's two, three.

Instructor: Put them both up there. We're going to debate that.

Kyle wrote the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ underneath his column vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. He then wrote his associated labels for the tall italicized N in a 2×3 matrix, as well as $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ underneath the second matrix (see Figure 2). In order to clarify to the reader these differences in labelling, we have included two other examples of student work below in Figure 3.

Kyle concluded his presentation to the class by explaining that there was a point of disagreement within his group about representation of inputs and outputs that arose as they were solving for the unknown values of the entries in the transformation matrix A :

Kyle: ... Our debate is about what these are [pointing to the column with the 0,3, written above and the 2,3 written below], not the process of solving ...

Instructor: Ok, so then the debate turned into, Kyle you can tell me if I'm interpreting you correctly, whether you thought of the thing over here as zero, three or two, three. [Kyle: Yeah, exactly.] Ok. Can you say some more about that?

Kyle: Um, we didn't work out everything if we used the second idea, we didn't work out all the equations yet. But when we used the first one, I mean you're

Figure 4 shows two panels of handwritten mathematical work. The left panel shows the derivation of matrix A. It starts with a vector $N = \begin{bmatrix} 0 & 2 & 0 \\ 3 & -3 & 3 \end{bmatrix}$ and another vector $N = \begin{bmatrix} 1 & 1 & 1 \\ 4 & -4 & 4 \end{bmatrix}$. A red arrow points from the second N to the first. Below, a matrix equation is set up: $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 3 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & -4 & 4 \end{bmatrix}$. This leads to a system of equations: $3A_1 = 1$, $2A_1 - 3A_2 = 1$, $3A_1 = 1$, $3A_3 = 4$, $2A_3 - 3A_4 = -4$, $3A_3 = 4$. The solution is $A_1 = \frac{1}{3}$, $A_2 = \frac{2}{3}$, $A_3 = \frac{4}{3}$, $A_4 = \frac{4}{3}$. The final matrix is $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{4}{3} \end{bmatrix}$. The right panel shows the matrix A being applied to vectors: $A \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $A \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Figure 4. Students set up matrix equations to solve for the values of matrix A.

basically, you're saying it's the same vector as the first one. So the two vertical lines would be the same vector. But if you grab the same vector, that's one spot. You'd basically have to add another, a shift, to get the vector to look like it does there. But then again, if you're at 2, 3, that's like an angular vector up. That's not a direct line upward. I don't know how to make it shift.

Instructor: Jake, does that connect to your guys's thinking at your table?

Jake: Uh, we didn't even consider it would be 2, 3. We thought it, we thought as, when the vector would go to the end, then we would think of that as the new origin. And then we would do the new vector from that end point.

In the ensuing discussion, several students weighed in, suggesting that the location of axes is relative. The instructor pointed out that if you do move the origin, then the labelling system does not distinguish the 0, 3 corresponding to the left segment of the N from what would be the 0, 3 corresponding to the right segment of the N. One group explained how they justified the top left point of the un-italicized Ns being called 0, 3 and 2, 3:

Lawson: We figured all those vectors add up to equal that, so if the answer works for all the individual vectors in the composition of that vector, it should work for that one too. We made an equation using the final points instead of the uh, points that add up to equal them. Or the vectors that add up to equal them.

Lawson's groupmate: We just used the line from the origin to that point as a linear combination of the rest.

The instructor encouraged students to keep a consistent origin throughout their work before sending them back to their groups to finish determining the transformation matrix A. Two groups explained in the ensuing whole class discussion how they set up a system of equations to find the entries of A as shown in their written work in Figure 4. Both used a labelling system more consistent with that of free vectors rather than labelling points on the N relative to a fixed origin. We conjecture that, because one can find matrix A using this approach, students did not experience a need to change their labelling system despite

the instructor's encouragement to choose the fixed origin system. We argue the fixed origin system is more compatible with a global view because it offers a way of viewing sets of inputs in a coordinated way.

5.2.2. Determining how many input–output pairs are needed to find the matrix A

After the class agreed that the matrix $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$ would yield the desired transformation, the instructor encouraged the class to engage in further mathematical investigations, asking if they had anything that they wanted to discuss about the solution approaches that the groups presented. Abraham asked, 'Do you need to put 0, 3 twice as a vector? Or like 1, 4? Does it need to repeat in the vector?' The instructor re-voiced this student question and posed it to the class, saying, 'Did we need all three to figure out A ? Could we have used just one? How many input–output pairs do we need to figure out A ?' In this way, the instructor leveraged Abraham's question towards a mathematical goal by reframing it in such a way that would allow the class to explore how many and what kind of input–output pairs are needed to uniquely determine the transformation matrix (assuming the standard basis). After discussing in their small groups, the instructor had Randall share his ideas with the class:

- Randall: I was looking at this matrix here and just the first two vectors span, so you don't need the third vector.
- Instructor: These two here? (points to columns 0,3 and 2, -3) [Randall: Yeah] Interesting. Say more about that.
- Randall: So we said that you only needed two to solve.
- Instructor: Ok, so I heard Randall make a claim. Because the first two vectors span \mathbb{R}^2 , we don't need a third to decide what A is. Something like that? Ok, well why? Why would that mean that you don't need a third?
- Randall: You can achieve all possible points in \mathbb{R}^2 with two vectors. Why would you need a third vector to get where you're going?
- Student: You're gonna achieve the same solution with those two. So like it doesn't help you to have the last one. And so you're going to do the same process you did with the first one.

Randall's claim that a set of inputs that span \mathbb{R}^2 is needed to determine A is correct, and another student helped justify this by stating that a repeated input–output pair doesn't give you any new information towards finding the solution. We also note that the examples of student reasoning we've highlighted in discussion of their work on this task give evidence to an increasingly coordinated global view – first being evidenced as Kyle showed the *set* of vectors his group chose to find the transformation matrix, then being refined in Abraham's question about whether all three vectors are needed and Randall's observation that you do not because the first two vectors span the input space.

The instructor concluded class by revisiting the approach shown in Figure 3(a) as an opportunity to formalize the product of matrices AB as A acting on the columns of B , because the class had only previously formalized the product of a matrix with a vector. Finally, homework was assigned that aimed to help students consider geometric interpretations of other linear transformations of the plane, such as stretching, skewing, reflecting, and rotating.

After class, a few students were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If an "E" is placed around the "N," the students argued over four different possible results for the transformed E's. Which choice below is correct, and why? If none of the four options are correct, what would the correct option be, and why?

Original:

Choice A:

Choice B:

Choice C:

Choice D:

Figure 5. Follow-up activity to the Italicizing N task.

In subsequent classroom teaching experiments, we have followed the Italicizing N Task with activities that ask students to investigate other transformations of the plane. Such follow-up activities press students to consider how the transformation defined by A affects the entire plane (the HLT's fourth learning goal), rather than just considering how A affects individual input-output pairs. In particular, it is important that students have opportunities to work on tasks that highlight how points that lie outside the first quadrant are transformed; this work can support students' further development of geometric interpretations of standard 2×2 transformation matrices. An example of such a task is shown in [Figure 5](#).

5.3 The Pat and Jamie task

The Pat and Jamie task (see [Table 1](#)) most explicitly supports the second learning goal of interpreting matrix multiplication as a composition of linear transformations. This is a follow-up to the Italicizing N Task, and it sets up a scenario in which two 'fellow students,'

Table 1. The Pat and Jamie task.

Last semester, two linear algebra students – Pat and Jamie – described their approach to the Italicizing N Task in the following way: 'In order to find the matrix A , we are going to find a matrix that makes the "N" taller (from 12-point to 16-point), find a matrix that italicizes the taller "N," and the combination of those will be the desired matrix A .'

1. Does their approach seem sensible to you? Why or why not?
2. Do you think their approach allowed them to find a matrix A ? If so, do you think it was the same matrix A we found this semester?
3. Try Pat and Jamie's approach. You should either: (a) come up with a matrix A by using their approach, or (b) be able to explicitly explain why this approach does not work.

Pat and Jamie, decide to solve the Italicizing N Task by first finding the matrix that makes the N taller, find the matrix that italicizes the taller N, and then combine them in a way to yield the correct transformation matrix. Students must first decide if the approach is valid and, if so, determine the matrices that represent the transformation via Pat and Jamie's approach. Note that in the problem statement, the order in which Pat and Jamie transform the N is not vague (they make it taller first and then italicize it), but the way in which to computationally accomplish this is purposefully left vague. The task positions students to engage in productive struggle with how to combine and symbolize one transformation followed by another and why that is sensible.

Two key issues emerged from students' work on this task: (a) using the outputs from the 'make taller' transformation as inputs in the 'italicize' transformation to find the correct matrix for the 'italicize' transformation; and (b) identifying and making sense of the order of matrix multiplication that matches Pat and Jamie's. In this section, we highlight the way in which the instructor leveraged coordination of local and global views to help students resolve these issues.

After the instructor had given students some time to work on the task in small groups, there was a brief whole class discussion in which students concluded that Pat and Jamie's approach seems sensible, and that it should yield the same matrix A as the previous task. In considering the ways in which Pat and Jamie's approach might be notated, the instructor encouraged students to name this matrix B for 'bigger' rather than T for 'taller,' pointing out that T had previously been used to refer to generic linear transformations.

As students then tried to find the matrix A via Pat and Jamie's approach, most were successful in constructing the correct matrix for the 'make taller' transformation $B = \begin{bmatrix} 1 & 0 \\ 0 & 4/3 \end{bmatrix}$. However, two main approaches were observed for constructing the 'italicize' or 'lean' matrix. Some students determined the correct 'lean' matrix, $L = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$. Others determined (incorrectly) that the matrix for the 'italicize' transformation in Pat and Jamie's approach is $L' = \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix}$, labelled in this paper as L' for ease of communication. Note that L' is the matrix that corresponds to the transformation of italicizing the original 12-pt font N, rather than the matrix that italicizes the taller 16-pt font N. When these students then tried to determine how to combine the matrices B and L' in such a way to equal matrix A , they found that $B L' = A$. This results in the aforementioned difficulty (b), which is not realizing that although $B L' = A$, it does not accurately reflect Pat and Jamie's process.

After students had worked on the Pat and Jamie task in small groups to try to find the 'make taller' and 'lean' matrices, the instructor initiated this whole class discussion by first clarifying that all groups had gotten the (correct) 'make taller' matrix and then asking students to share the matrices they found for the 'lean' transformation. The instructor asked two groups who had gotten the correct lean matrix (labelled L in the preceding paragraph) to explain their approach to the class, as they had gotten a different lean matrix than most of the other groups. The first group focused their explanation on how they computed the lean matrix L , whereas the second group focused their explanation on how they combined that matrix L with B to yield A . The second group pointed out that they tried adding and multiplying, and found that $LB = A$ but was unsure why it made sense to combine the matrices in this way. The instructor used this opportunity to generate a need for consensus, pointing out that both proposed 'lean' matrices could be combined (through multiplication) with the two proposed 'make taller' matrices to yield the correct transformation



Figure 6. A transformational record extending Eric's explanation.

matrix A that students had found in the previous task, i.e. that $\begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4/3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. At the same time, these two approaches had different implications for the order in which the transformations were applied to the original N .

At this point, the instructor sent students back to their small groups for a few minutes to decide which approach was consistent with what Pat and Jamie did and why that order of multiplication made sense. The instructor then asked one student, 'Eric,' to share what his group had discussed:

Eric: Judging by the instructions that it gives us, it says, by their approach, find the matrix that italicizes the larger N , would be how they did it [pointing to the group who had presented the correct 'lean' matrix], because the other one italicizes the smaller N . So if you're going by the directions, the way they did it would be the first method.

Instructor: By the way who did it?

Eric: Jamie and Pat.

Instructor: Ok, so it sounds like you're saying, we know we italicized the larger one, so it has to be this one. But the order, still not sure?

Eric: Uh, the orders matter.

Here Eric offered a correct argument about which of the approaches Pat and Jamie took, but said he was unsure as to why the order of multiplication was sensible. Note that Eric's argument drew on a global view (organized around the mapping of the N as a whole, rather than mapping individual inputs to outputs). The instructor then drew on a local view to restate Eric's argument in a way that would help clarify the reasoning behind the order of multiplication. The instructor redrew a diagram that had been used throughout the class to orient whole class discussion around students' work on this task, and used this to trace a generic input vector \mathbf{x} through the series of transformations while linking it to Eric's explanation that matrix L should italicize the taller letter 'N' (see Figure 6). She showed the input vector \mathbf{x} , being mapped to $B\mathbf{x}$ (which she relabelled as \mathbf{y}) on the taller N , and that vector \mathbf{y} being mapped to $L\mathbf{y}$ on the final, italicized N – pointing out then that $L\mathbf{y} = L(B\mathbf{x}) = LB\mathbf{x}$. This pedagogical move is consistent with what Rasmussen and Marrongelle [41] characterize as a transformational record – wherein an instructor provides symbolization for a student's idea in a way that helps draw on that student's thinking to advance the mathematical agenda of the class.

We highlight this episode as significant because it illustrates (a) the ways in which coordination of global and local views can foster an intuitive notion of function composition in the context of linear transformations, and (b) the role of the instructor in connecting

Table 2. The getting back to N task.

Regarding the Italicizing N Task, complete the following: Find a matrix C that will transform the letter on the right back into the letter on the left.

1. Find C using either your method or one of your classmates' methods for finding A .
 2. Find C using Pat and Jamie's method for finding A .
-

to student thinking as she moves her mathematical agenda forward. As a member of the mathematics community, the instructor serves as a broker [42] between students' authentic mathematical activity and the terminology and notation commonly used in the mathematics community.

5.4. The getting back to N task

The last main task in the instructional sequence is the Getting Back to N task (see Table 2). The primary learning goal associated with this task is the third, students developing the idea of an inverse as 'undoing' the original transformation. It is also intended to further the second learning goal, reasoning about matrix multiplication as a composition of linear transformations, through exploring the inverse of a composition of linear transformations. This task asks students to find a matrix C that will transform the 16-pt italicized N back to the original N. Furthermore, this task asks students to determine the matrix C in two ways: by using a single transformation and by using Pat and Jamie's method (i.e. by using two transformations to 'undo' what Pat and Jamie did). In this task, we highlight the ways in which some students' work foregrounds the local view while other students' work foregrounds the global view. We argue that both views are important and describe how the instructor worked to elicit and help students coordinate their thinking about both views.

In working on the first prompt of finding the matrix C , the matrix that undoes the transformation in a single step, many students drew on their strategies from the previous tasks to determine C by creating a matrix equation that coordinates particular input–output pairs from the tall, italicized N to the original N. The instructor highlighted the relationship between these input–output pairs and those from the original Italicizing N task; namely, that the inputs in the Getting Back to N task served as the outputs in the Italicizing N task and vice versa. This emphasis on the role of the components of matrix equations such as $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{b} = \mathbf{x}$ laid the groundwork for subsequently labelling the matrix C as the inverse of matrix A . In our example, the instructor drew on students' reasoning that the transformation defined by C 'undoes' the effect of the transformation defined by A , and supported this reasoning by introducing a notational shift $C = U_A$, with the explanation that U_A stands for the matrix that 'undoes A .' Given that the instructor had not, at this stage, formalized C as the inverse of A , the symbol used to notate this relationship (U_A) was tied to students' ways of thinking and symbolizing.

When working on the second prompt of determining the matrix C via Pat and Jamie's method, the students were faced again with not only determining the matrix transformations for the constituent parts, but also with determining how to combine those matrices in a sensible manner. Here, the instructor similarly connected to the class's previously used labels (e.g. the letter B to label the matrix that made the N bigger, L for the matrix that made the N lean, and A is the original transformation matrix from the Italicizing N task), with U_L , U_B , and U_A , corresponding to the transformations that 'undo' the original transformations defined by L , B , and A , respectively. Also note that the notation in Figure 7 is layered

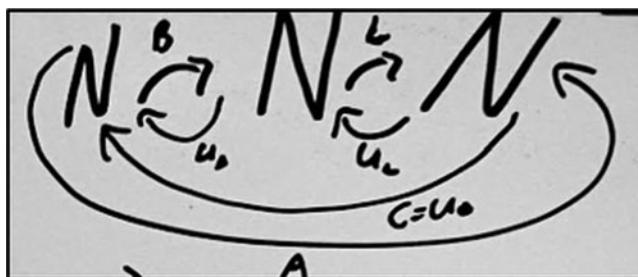


Figure 7. Negotiation of meaning for the ‘undoing’ matrices U_B , U_L and U_A .

Figure 8. Student work on the Getting Back to N task.

on the symbolic presentations developed in the original Pat and Jamie task (see Table 1), in order to further connect to previous work and discussions.

Students subsequently drew on the notation in Figure 7 in calculating the numerical values for the components of U_L , U_B , and U_A . The example of student work in Figure 8(a) illustrates how one group notated the idea that that matrix multiplication for a composition of functions is constructed right to left, with the matrix U_L for the first transformation being multiplied on the left by the matrix U_B for the subsequent transformation. The second example of student work (see Figure 8(b)) shows a student’s notation illustrating that A composed with its ‘undoing’ matrix U_A in either order leaves the input vector unchanged. We argue that the first group’s work foregrounds a global view whereas the second group’s work foregrounds a local view by emphasizing what happens to an individual input. These approaches highlight important aspects of the mathematics – both how U_B and U_L matrices can be combined to yield the matrix that undoes the transformation in a single step (Figure 8(a)), and how the composition of a matrix with its ‘undoing’ matrix maps an element back to itself (Figure 8(b)).

The instructor used the student work shown in Figure 8(b) to facilitate a conversation about interpreting $U_A A$ and $A U_A$ as transformations that have the action of ‘doing nothing’ to any given input vector, leading to symbolizing the matrix for the ‘do nothing’ transformation by the letter I ; that is, $(U_A A)x = Ix = x$ and $(A U_A)x = Ix = x$. As a member of the classroom community and the mathematics community, the instructor acts as a broker between them by connecting the class’s work with the formal definitions of inverse for both linear transformation (i.e. a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible given that there exists a transformation $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $T(S(x)) = x$ and $S(T(x)) = x$ for all x in \mathbb{R}^n) and matrix multiplication (i.e. an $n \times n$ matrix A is invertible given that there exists an $n \times n$ matrix C such that $AC = I$ and $CA = I$).

6. Discussion

In this paper we offer a contribution to the growing body of research in the teaching and learning of linear algebra by presenting an HLT for understanding matrices as linear transformations. In particular, we highlighted the ways in which students draw on local and global views across a sequence of tasks, and the role of the instructor in supporting students to coordinate these views to support their learning about a transformation interpretation of the matrix equation $A\mathbf{x} = \mathbf{b}$ and to extend this to make sense of composition and inverses.

6.1. Student reasoning: local and global views

We argue that a central contribution of this work is the articulation of the importance and role of coordinating local and global interpretations as students develop transformation views of matrix multiplication. This occurs in two ways: students come to reason about multiplication by a single matrix as a linear transformation, and students come to reason about composing and inverting such transformations. By detailing this HLT as a possible route for developing students' understanding of matrices as linear transformations and considering what aspects of this route are likely to generalize across instructional settings, we more broadly contribute to research on student reasoning about functions in higher dimensional settings.

In particular, we highlight students' initially local transformation interpretations, which focus on coordinating an input vector with an output vector (where the output vector is produced by multiplying an input vector by the transformation matrix). In the context of the task sequence, student reasoning related to the development of this interpretation involved making choices about how to label inputs and outputs, as well as considering the dimensions of a matrix that would yield the desired transformation. This leads to a push toward more global views, especially as students begin to wrestle with the question about the number of input/output pairs needed to determine the transformation. In turn, this also carries inherent conceptual connections to ideas about span, linear independence, bases, and linearity.

A lens that has often been brought to making sense of shifting mathematical reasoning, particularly with regard to functions, is Sfard's notion of process and object views of mathematical ideas [32]. Sfard argues that many important mathematical ideas develop through a process – object shift, both when viewed from a broad historical perspective and when viewed from the perspective of individual cognitive development. Specifically, many notions are first conceived of as a process. This process, when reified, can be seen as an object which can itself be the subject of study and something that can be operated on.

In our study, students' early work in the task sequence is more indicative of process views of matrix transformations, and their later work to combine (compose) and undo (invert) these transformations presses them toward object views of matrix transformations. We argue that the shift from process to object views of matrices as linear transformations is distinct from but importantly related to students' coordination of local and global views of matrix transformations. For instance, as highlighted in our examples, the local view can helpfully inform the rationale for computational order and notational representations for composition and inverses. However, there is a need to shift away from a local view that focuses on individual inputs and outputs in order to consider how two transformations can

be combined or inverted, as this involves thinking about how the matrix transformations themselves interact with one another (and in fact requires associativity of matrix multiplication). The global view, on the other hand, may be less helpful for thinking about how to combine transformations, but is important for supporting how students come to think about the transformation affecting the space in a coordinated way – which points to important aspects of the structure of \mathbb{R}^n as a vector space (in particular, a space that is closed under linear combinations) and matrix transformations as linear mappings which nicely preserve that structure.

6.2. Instructor role

The instructor's role is inherently intertwined with the development of students' reasoning documented in this paper. We highlight two aspects of the role of the instructor: the role of the instructor as broadly aligned with literature on inquiry-oriented instruction in undergraduate mathematics, and the role the instructor played in shaping the trajectory of student reasoning in this particular HLT implementation.

The literature details multiple aspects of the role instructors play in inquiry-oriented mathematics classrooms: posing tasks to students that support learning goals, providing students with opportunities to collaborate with peers, inquiring into student reasoning, and facilitating discussions in which students explain and justify their reasoning in ways that advance the mathematical agenda of the class [e.g. 38]. An example of the latter from our data is when the instructor identified the mathematical value in Abraham's question about whether a particular vector needed to be considered to determine the transformation matrix A . The instructor reframed this as a broader question about how many input–output pairs are needed to determine the matrix A . This opened an opportunity for Randall to share his thinking that a spanning set is what is needed.

When implementing an HLT, instructors must also take into account how to support the *development* of students' reasoning relative to a sequence of tasks and set of mathematical learning goals. The choices instructors make as they respond to student contributions importantly shape the development of student reasoning. To illustrate this, we highlighted several examples of the instructor's decisions about what aspects of student reasoning to foreground in classroom mathematical discussions, as well as what and when to 'tell' that impacted the trajectory of students' work. For example, the instructor chose to tell that the transformation matrix was 2×2 after students had an opportunity to form some initial ideas about this question. Another example is how the instructor framed the discussion of the two approaches to the Pat and Jamie task. She did not merely ask students to share their approaches in sequence, but rather she framed the purpose of the discussion to resolve a very particular point of disagreement about the order of multiplication relevant to the mathematical goal of reasoning about composition. Lastly, we highlighted the use of an organizing diagram to facilitate classroom discussion around the task sequence. The use of this diagram fostered students' coordination of global and local views and helped students make sense of computational and notational aspects of compositions and inverses of matrix transformations. Zandieh and colleagues refer to this as a unifying diagram and argue it is a form of symbolizing that instructors can engage in to support students in communicating their reasoning in inquiry-oriented classrooms [43]. Taken together, the examples highlight

the integral role an instructor plays in supporting the trajectory of student reasoning in any given enactment of an HLT.

The HLT presented in this paper offers insight into our research team's ongoing work. We continue to learn how this instructional sequence plays out in different classrooms with different groups of students and make adjustments accordingly, such as by developing supplemental tasks and instructional strategies to account for variation in students' prior knowledge (see <http://iola.math.vt.edu> for more information).

7. Limitations

The findings of this study are drawn from close analysis of data taken from a single classroom teaching experiment. Other classroom implementations of this sequence of activities naturally give rise to variations in student approaches and classroom discussions. This limits the generalizability of this study in some ways. However, as previously mentioned, this study aims to document a *possible* path for the development of students' reasoning about matrices as linear transformations. We argue that central aspects of student learning are likely to generalize and are worth considering for instructors and curriculum developers concerned with this topic, even if their instructional approach is different from the one offered in this paper. Future work is needed to examine what aspects of this work generalize to the development of students' reasoning about generalized vector spaces and how students come to reason about the role of linearity in understanding the structure of vector spaces and mappings between them.

8. Conclusion

The analysis presented in this paper has the potential to inform the work of mathematics education researchers, mathematics instructors, and curriculum developers. In documenting a possible path for supporting students' development of a transformation view of matrix multiplication, we identify important aspects of student reasoning and the role the instructor plays in supporting the development of that reasoning.

By detailing the importance of coordinating local and global views of a transformation interpretation of matrix multiplication and relating that coordination to research on process and object conceptions of mathematical notions, we offer a contribution to basic research on student reasoning in introductory linear algebra. By identifying important aspects of the instructor's role in the development of student reasoning, we contribute to research that links instruction to student learning. Taken together, this lays a valuable foundation that can help instructors interpret student reasoning, that can help curriculum developers identify appropriate learning goals, and that can help mathematics education researchers better understand the development of student reasoning in undergraduate linear algebra.

Acknowledgments

We give special thanks to Dr. Chris Rasmussen for his guidance and feedback on this work. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the Division of Undergraduate Education and Division of Research on Learning in Formal and Informal Settings, National Science Foundation [grant numbers DRL 0634099, 0634074, DUE 1245673, 1245796, 1246083, and 1431393].

References

- [1] Gravemeijer K, Bowers J, Stephan M. A hypothetical learning trajectory in measurement and flexible arithmetic. *J Res Math Educ.* 2003;12:51–66.
- [2] Simon MA. Reconstructing mathematics pedagogy from a constructivist perspective. *J Res Math Educ.* 1995;26:114–145
- [3] Larsen S. Reinventing the concepts of groups and isomorphisms: the case of Jessica and Sandra. *J Math Behav.* 2009;28:119–137.
- [4] Bagley S, Rasmussen C, Zandieh M. Inverse, composition, and identity: the case of function and linear transformation. *J Math Behav.* 2015;37:36–47.
- [5] Oehrtman MC, Carlson MP, Thompson PW. Foundational reasoning abilities that promote coherence in students' understandings of function. In: Carlson MP, Rasmussen C, editors. *Making the connection: research and practice in undergraduate mathematics.* Washington (DC): MAA; 2008. p. 150–171.
- [6] Zandieh M., Ellis J., Rasmussen C. A characterization of a unified notion of mathematical function: the case of high school function and linear transformation. *Educ Stud Math.* Forthcoming. doi:10.1007/s10649-016-9737-0
- [7] Cobb P. Conducting teaching experiments in collaboration with instructors. In: Kelly AE, Lesh RA, editors. *Handbook of research design in mathematics and science education.* Mahwah: Lawrence Erlbaum; 2000. p. 307–330.
- [8] Cobb P, Confrey J, diSessa A, et al. Design experiments in educational research. *Ed Researcher.* 2003;32:9–13.
- [9] Nickerson SD, Whitacre I. A local instruction theory for the development of number sense. *Math Thin Learn.* 2010;12:227–252.
- [10] Rasmussen C, Keynes M. Lines of eigenvectors and solutions to systems of linear differential equations. *PRIMUS.* 2003; 13: 308–320.
- [11] Clements DH, Sarama J. Learning trajectories in mathematics education. *Math Think Learn.* 2004;6:81–89.
- [12] Stephan M, Akyuz D. A proposed instructional theory for integer addition and subtraction. *J Res Math Educ.* 2012; 43: 428–464.
- [13] Britton S, Henderson J. Linear algebra revisited: an attempt to understand students' conceptual difficulties. *Int J Math Ed Sci Technol.* 2009; 40: 963–974.
- [14] Carlson D. Teaching linear algebra: must the fog always roll in? *College Math J.* 1993;24:29–40.
- [15] Dorier J-L, Robert A, Robinet J, et al. The obstacle of formalism in linear algebra: a variety of studies from 1987 until 1995. In: Doreir J-L, editor. *On the teaching of linear algebra.* Dordrecht: Kluwer; 2000. p. 85–124.
- [16] Harel G. Learning and teaching linear algebra: difficulties and an alternative approach to visualizing concepts and processes. *Focus Learn ProblMath.* 1989;11:139–148.
- [17] Hillel J. Modes of description and the problem of representation in linear algebra. In: Doreir J-L, editor. *On the teaching of linear algebra.* Dordrecht: Kluwer; 2000. p. 191–207.
- [18] Sierpinska A. On some aspects of students' thinking in linear algebra. In: Doreir J-L, editor. *On the teaching of linear algebra.* Dordrecht: Kluwer; 2000. p. 209–246.
- [19] Stewart S, Thomas M. Student learning of basis, span and linear independence in linear algebra. *Int J Math Educ Sci Technol.* 2010;41:173–188.

- [20] Wawro M, Sweeney G, Rabin J. Subspace in linear algebra: investigating students' concept images and interactions with the formal definition. *Educ Studies Math.* 2011;78:1–19.
- [21] Dogan-Dunlap H. Linear algebra students' modes of reasoning: geometric representations. *Linear Algebra Appl.* 2010;432:2141–2159.
- [22] Gueudet-Chartier G. Should we teach linear algebra through geometry? *Linear Algebra Appl.* 2004;379:491–501.
- [23] Harel G. Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. *Int J Math Educ Sci Technol.* 1990; 21: 387–392.
- [24] Harel G, Kaput J. The role of conceptual entities in building advanced mathematical concepts and their symbols. In: Tall D, editor. *Advanced mathematical thinking.* Dordrecht: Ridell; 1991. p. 82–94.
- [25] Larson C, Zandieh M. Three interpretations of the matrix equation $Ax = b$. *For the Learn Math.* 2013;33:11–17.
- [26] Carlson D, Johnson CR, Lay DC, Porter AD. The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *College Math J.* 1993; 41–46.
- [27] Carlson MP. A cross-sectional investigation of the development of the function concept. *CBMS Issues Math Educ: Research in Collegiate Math Educ III.* 1998;7:114–162.
- [28] Jacobs S, Trigueros M. On developing a rich conception of variable. In: Carlson MP, Rasmussen C, editors. *Making the connection: research and practice in undergraduate mathematics.* Washington (DC): MAA; 2008. p. 3–13.
- [29] Freudenthal H. *Didactical phenomenology of mathematical structures.* Dordrecht: Reidel; 1983.
- [30] Cobb P. Learning from distributed theories of intelligence. In: Yackel E, Gravemeijer K, & Sfard A, editors. *A journey into mathematics education research: insights from the work of Paul Cobb.* New York: Springer; 2011. p. 85–105.
- [31] Gravemeijer, K. How emergent models may foster the constitution of formal mathematics. *Math Think Learn.* 1999;1:155–177.
- [32] Sfard A. On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educ Stud Math.* 1991;22:1–36.
- [33] Simon MA, Tzur R. Explicating the role of the mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory. *Math Think Learn.* 2004;6:91–104.
- [34] Steffe L. On the construction of learning trajectories of children: the case of commensurate fraction. *Math Think Learn.* 2004;6:129–162.
- [35] Cobb P, Stephan M, McClain K, et al. Participating in classroom mathematical practices. *J Learn Sci.* 2001;10:113–163.
- [36] Stephan M, Rasmussen C. Classroom mathematical practices in differential equations. *J Math Behav.* 2002;21:459–490.
- [37] Cobb P, Yackel E. Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Ed Psychologist.* 1996; 31: 175–190.
- [38] Rasmussen C, Kwon O. An inquiry oriented approach to undergraduate mathematics. *J Math Behav.* 2007;26:189–194.
- [39] Wawro M, Rasmussen C, Zandieh M, et al. An inquiry-oriented approach to span and linear independence: the case of the Magic Carpet Ride sequence. *PRIMUS.* 2012;22:577–599.
- [40] Lay, DC. *Linear algebra and its applications.* 3rd ed. Reading (MA): Addison-Wesley; 2003.
- [41] Rasmussen C, Marrongelle K. Pedagogical content tools: Integrating student reasoning and mathematics into instruction. *J Res Math Educ.* 2006;37:388–420.
- [42] Rasmussen C, Zandieh M, Wawro M. How do you know which way the arrows go? The emergence and brokering of a classroom mathematics practice. In: Roth W-M, editor. *Mathematical representation at the interface of body and culture.* Charlotte: Information Age; 2009. p. 171–218.
- [43] Zandieh M., Wawro M., Rasmussen C. An example of inquiry in linear algebra: the roles of symbolizing and brokering. *PRIMUS.* 2017;27:96–124.